Digital Signal processing

Syllabus

1. Introduction.
   1.1 Discuss Signals, Systems & signal processing.
      1.1.1 Explain basic element of a digital signal processing System.
      1.1.2 compare the advantages of digital Signal processing over analog signal processing. ~~classify Signals~~.
   1.2 classify Signals
      1.2.1 multi channel & Multi dimensional Signals.
      1.2.2 continuous time Verses Discrete - time signals.
      1.2.3 continuous valued Verses Discrete - valued Signals.
   1.3 Discuss the concept of frequency in continuous time & discrete time Signals.
      1.3.1 continuous - time sinusoidal Signals
      1.3.2 Discrete - time sinusoidal signals
      1.3.3 Harmonically related complex exponential.
   1.4 Discuss Analog to Digital & Digital to Analog conversion & explain the following.
      1.4.1 Sampling of Analog Signal.
      1.4.2 The Sampling theorem.
      1.4.3 Quantization of continuous amplitude signals.
      1.4.4 coding of quantized sample.
      1.4.5 Digital to analog conversion.
      1.4.6 Analysis of digital systems Signals Vs. discrete time Signals.
2. Discrete Time Signals & System.
   2.1 State and explain discrete time signals.
      2.1.1 Discuss some elementary discrete time signals
      2.1.2 classify discrete time signal.
      2.1.3 Discuss simple manipulation of discrete time signal.
   2.2 Discuss discrete time system.
      2.2.1 Describe input - output system.
      2.2.2 Draw block diagram of discrete time ~~Signal~~ System.
      2.2.3 classify discrete time system.
      2.2.4 Discuss interconnection of discrete - time System.
   2.3 Discuss discrete time time - invariant system.
      2.3.1 Discuss different technique for the analysis of linear system.
      2.3.2 Discuss the resolution of a discrete time Signal in to impulse.
      2.3.3 Discuss the response of LTI System to arbitary I/Ps using convolution theorem.

2.3.4 Explain the properties of convolution & interconnection of LTI system.

2.3.5 Study system with finite duration and infinite duration impulse response.

2.4 Discuss discrete time system described by difference system.

2.4.1 Explain recursive & non-recursive discrete time system.

2.4.2 Determine the impulse response of linear time invariant recursive system.

3. The Z-transform & it's application to the analysis of LTI system.

3.1 Discuss Z-transform & it's application to LTI system.

3.1.1 State & explain direct Z-transform.

3.1.2 State & explain inverse Z-transform

3.2 Discuss various properties of Z-transform.

3.3 Discuss rational Z-transform.

3.3.1 Explain poles & zeros.

3.3.2 Determine pole location time domain behaviour for causal signals.

3.3.3 Describe the system function of a linear time invariant system.

3.4 Discuss inverse Z transform.

3.4.1 Determine inverse Z-transform by partial fraction expansion.

4. Discuss fourier transform & it's application propenties.

4.1 Discuss discrete fourier transform.

4.2 Determine frequency domain sampling & reconstruction of discrete time signals.

4.3 State & explain Discrete time fourier transform (DTFT)

4.4 State & explain Discrete fourier transform (DFT)

4.5 Compute DFT as a linear transformation.

4.6 Relate DFT to other transform.

4.7 Discuss the property of the DFT.

4.8 Explain multiplication of two DFT & circular convolution.

5. Fast fourier Transform Algorithm & Digital Filters.

5.1 Compute DFT & FFT algorithm. (Small problems)

5.3 Discuss the radix-2 algorithm.

5.2 Explain direct computation of DFT.

5.4 Introduction to digital filters (FIR filters)

5.5 Introduction to DSP architecture, familarsation of different types of processor.

Text Book

1. Digital Signal processing principles algorithm & application by J.G. Proakis & Dimities G. Manolakis, Pearson.
2. Digital Signal processing by Ramesh babu

Chapter - 1

1. Introduction

1.1 a) **Signal**

A Signal is defind as any physical quantity varies with time, space or any other independent variable and variables.

mathemattically, we describe a signal as a function of one or more independent variables. for example the function

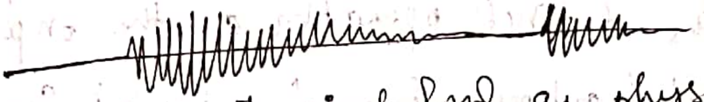
$S_1(t) = 5t$

$S_2(t) = 20t^2$

t = independent variable of two signal.

Ex- for independent variable x & y -

$s(x, y) = 3x + 2xy + 10y^2$



b) System:- ~~Aoph~~ it is defind as physical device that perform an operation on a signal. ~~for~~

Ex- a filter used to reduce the noise & interference corrupting a desired information.

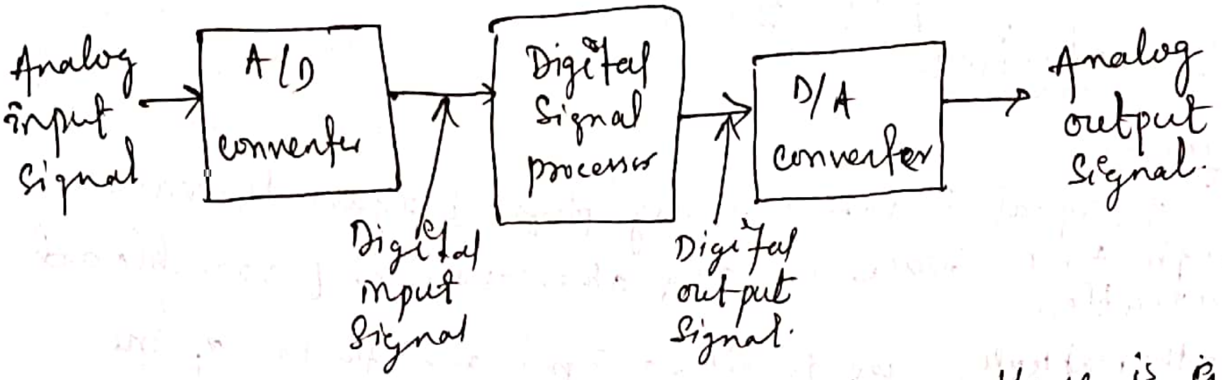
c) **Signal processing**

when we pass a signal through a system as in filtering we say that we have processed the signal.

1.1.1 Explain basic element of a digital signal processing system.

→ most of the signals encountered in science & engineering are analog in nature. This is the signal are functions of a contineous variable, such as time or space, & usually take on values in a contineous range. Such signals may be processed directly by appropriate analog

Digital signal processing provides an alternative method for processing the analog signal.



→ To perform the processing digitally, there is a need for an interface betn the analog signal and the digital processor. This interface is called as analog-to-digital (A/D) converter.

→ The digital signal processor may be a large programmable digital computer or a small microprocessor programmed to perform the desired operation on the input signal.

→ where the digital output from the digital signal processor is to be given to the user in analog form ~~the~~ such as in speech communication, we must provide another interface from the digital domain to the analog domain. It is called digital to analog (D/A) converter.

1.1.2 Compare the advantage of digital signal processing over analog signal processing.

→ Digital programmable system allows flexibility in reconfiguring the digital signal processing operation simple by changing the program.

→ Reconfiguration of an analog system usually implies a redesign of the hardware followed by testing & verification to see that it operates properly.

→ Accuracy considerations also play important role in determining the form of the signal processor.

→ Tolerance in analog circuit components makes extremely difficult for the system designer to control accuracy of an analog signal processing system.

→ Digital provide much better accuracy of an requirement.

→ Digital Signal are easily stored on magnetic media (tape or disc) without deteriorating or loss of Signal fidelity. beyond the A/D conversion.

→ Digital Signal processing method also allows for the implementation of more sophisticated Signal processing algorithm.

→ Digital Signal processing is cheaper than analog.

→ Speed of Digital Signal processing higher than analog.

→ ~~Fast response~~ Digital Signal processing fast response with large bandwidth.

1.2 classification of Signals

1.2.1 multichannel & multidimensional Signals

→ A Signal is described by a function of one or more independent variable. the value of the function can real - valued Scalar quantity, a complex valued quantity, or perhaps a vector.

→ vector $S_3(t)$

$$S_3(t) = \begin{bmatrix} S_1(t) \\ S_2(t) \\ S_3(t) \end{bmatrix}$$

we refer to such a vector of signals as a multi channel Signal. ECG have 3-leads means
↳ Electrocardiography
3-channel Signal & ECG have 12 - leads mean 12 channels. that is called multichannel.

→ Multidimensional → if the Signal is a function of a Single independent variable, the Signal is called a ~~AA~~ one - dimensional Signal, if is M no. of independent variable is called M - dimensional Signal.

$$I(x, y, t)] = \begin{bmatrix} I_r(x, y, t) \\ I_g(x, y, t) \\ I_b(x, y, t) \end{bmatrix}$$

This is funcfiln of color TV·picture.

Channel are (red, green, blue)

Dimension (x, y, t)

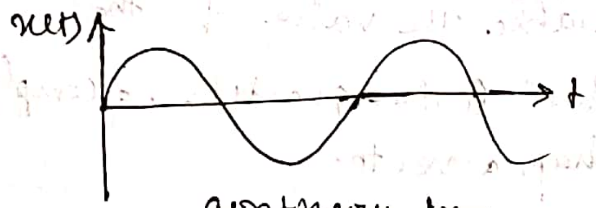
1.2.2 Continuous - Time Versus Discrete - Time Signals

Continuous - time Signals

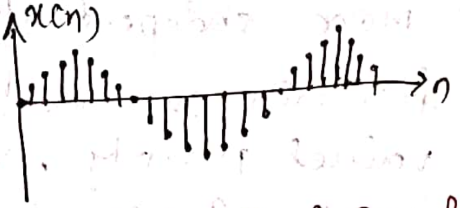
The Signals that are defined for every instant of time are known as continuous - time signal. They are denoted by x(t).

Discrete - time Signals

The Signals that are defined at discrete instants of time are known as discrete - time Signals. The discrete - time Signals are continuous in amplitude and discrete in time. They are denoted by x(n). & define at certain specific values of time.



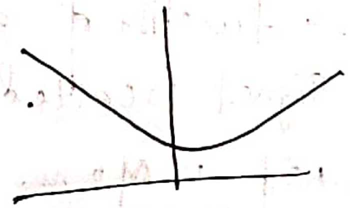
continneous time



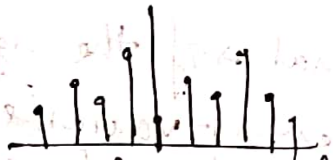
1.2.3 Continuous - valued Versus Discrete valued Signals

→ If a Signal takes on all possible values on a finite or an infinite range, it is said to be a continuous - valued Signals.

→ If the Signal takes on values from a finite set of possible values it is said to be a discrete valued Signal.



contineous value

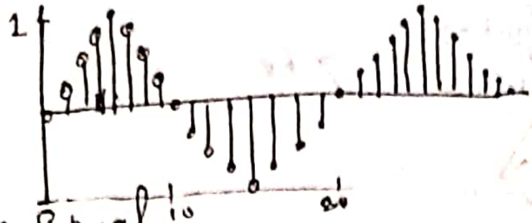


Discreek values

Deterministic Versus Random Signal

Deterministic Signal:- A Signal ~~[illegible]~~ exhibiting no uncertainty of value at any instant of time. if instaneous value can be accurately predicted by mathematical equation. one such signal

$$x_1(n) = \sin(0.1\pi n)$$



Random Signal:-

A Random Signal is a Signal characterized by the uncertainty before it's actual occurance. Ex Noise.

1.3 The concept of frequency in continuous-time & Discrete-Time Signal

1.3.1 continuous-Time Sinusoidal Signals

A simple harmonic oscillating is mathematically described by the following continous time sinusoidal Signal.

$$x_a(t) = A\cos(\omega t + \theta), \quad -\infty < t < \infty$$

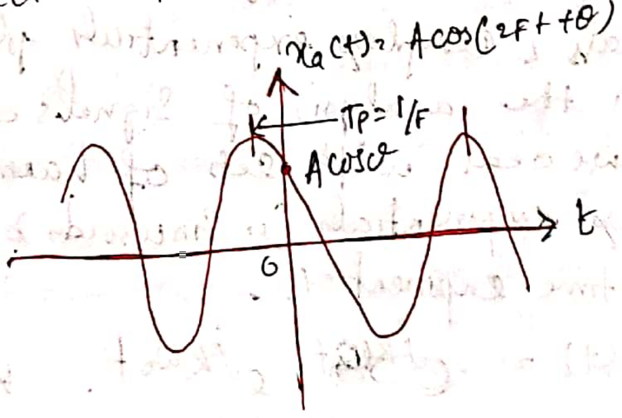
A = amplitude

ω = frequency radian per second.

θ = phase radians.

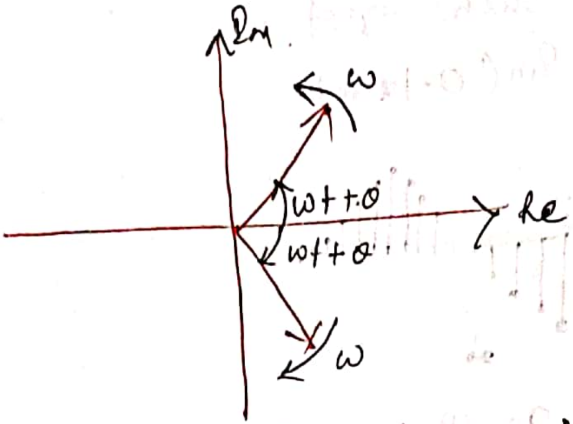
$\omega = 2\pi F$

$$x_a(t) = A\cos(2\pi F t + \theta), \quad -\infty < t < \infty$$



T_p = fundamenta time periode.

$f = \frac{1}{T_p}$ $\boxed{T_p = \frac{2\pi}{\omega}}$



A positive frequency corresponds to counter clock wise uniform angular motion, a negative frequency simply correspond to clock wise angular motion

1.3.2 Discrete-Time Sinusoidal Signals

A discrete-time Sinusoidal Signal may be expressed as

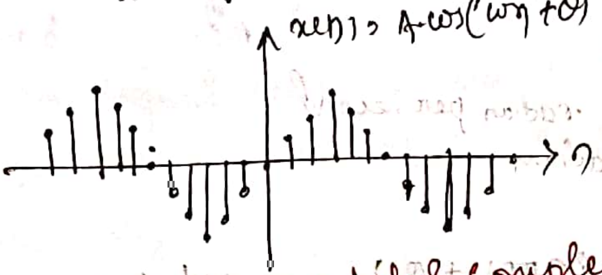
$$x(n) = A\cos(\omega n + \theta), \quad -\infty < n < \infty$$

A = amplitude

ω = frequency rad/sec

θ = phase.

$\omega = 2\pi f$



1.3.3 Harmonically Related complex exponentials

→ Sinusoidal Signals & complex exponentials play a major role in the analysis of Signals and System. In some cases we deal with sets of harmonically related complex exponentials (or sinusoids).

continuous-time exponential -

$$S_k(t) = ~~e^{jk\omega_0 t}~~ \; e^{jk\omega_0 t}. \quad k = 0, \pm1, \pm2$$

Discrete - time exponential

$$s_k(n) = e^{j2\pi k f_0 n} \quad k=0, \pm1, \pm2$$

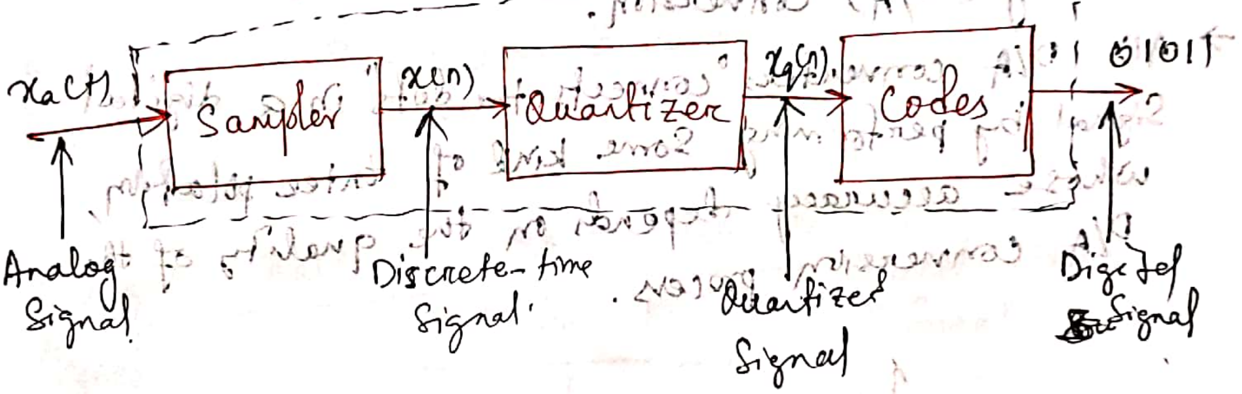
1.4 Analog - to - Digital & Digital - to - Analog Conversion

→ Most signals of practical interest, such as speech, biological signals, seismic signals, radar signals, sonar signals & various communication signals such as audio & video signals are analog.

→ To process analog signals by digital means, it is first neccisary to convert them in to digital form; that is to convert them to a sequence of number having finite precision.

→ This procedure is called analog to digital (A/D) conversion & the corresponding devices are called 'A/D' convents (ADCs).

→ we view A/D conversion as a three step process



1. Sampling → This is the conversion of a continuous time signal in to a discrete signal obtained by taking "samples" of the continuous - time signal at discrete time instants. Thus if $x_a(t)$ is the input to the sampler, the output is $x_a(nT) \equiv x(n)$ where T is called the sampling interval.

2. Quantization :- This is the conversion of a discrete-time continuous - valued signal in to a discrete - time, discrete valued (digital) signal.

→ The value of each signal sample is represented by a value selected from a finite set of possible value. The difference betn the quantized sample $x(n)$ & the quantized output $x_q(n)$ is called the quantization error..

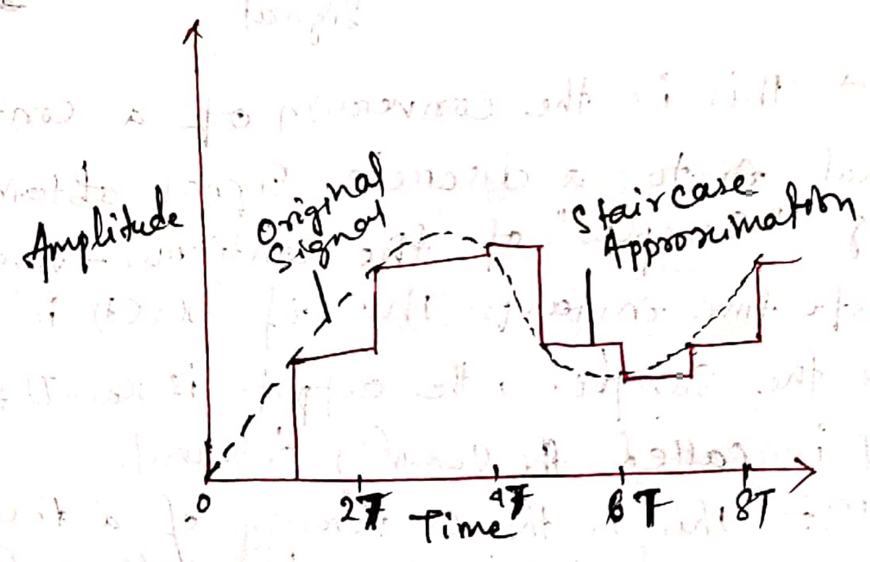
3- Coding :- In the coding process each discrete value $x_q(n)$ is represented by a ~~bit~~ b-bit binary sequence.

→ Although we model the A/D converter as a sampler followed by a quantizer and coder, in practice the A/D conversion is performed by a single device that takes $x_a(t)$ & produces a binary-code number.

→ the operation of sampling & quantization can be performed in either order but in practice, sampling is always performed before quantization.

→ The process of converting a digital signal ~~in~~ to an analog signal is known as digital to analog (D/A) conversion.

→ All D/A converters "connects the dots" in a digital signal by performing some kind of interpolation, whose accuracy depends on the quality of the D/A conversion process.



→ A simple form of D/A conversion is called a zero-order hold or a staircase approximation.

→ Sampling and quantization are treated as we demonstrate that sampling does not result in a loss of information nor does it introduce distortion in the signal if the signal bandwidth is finite.

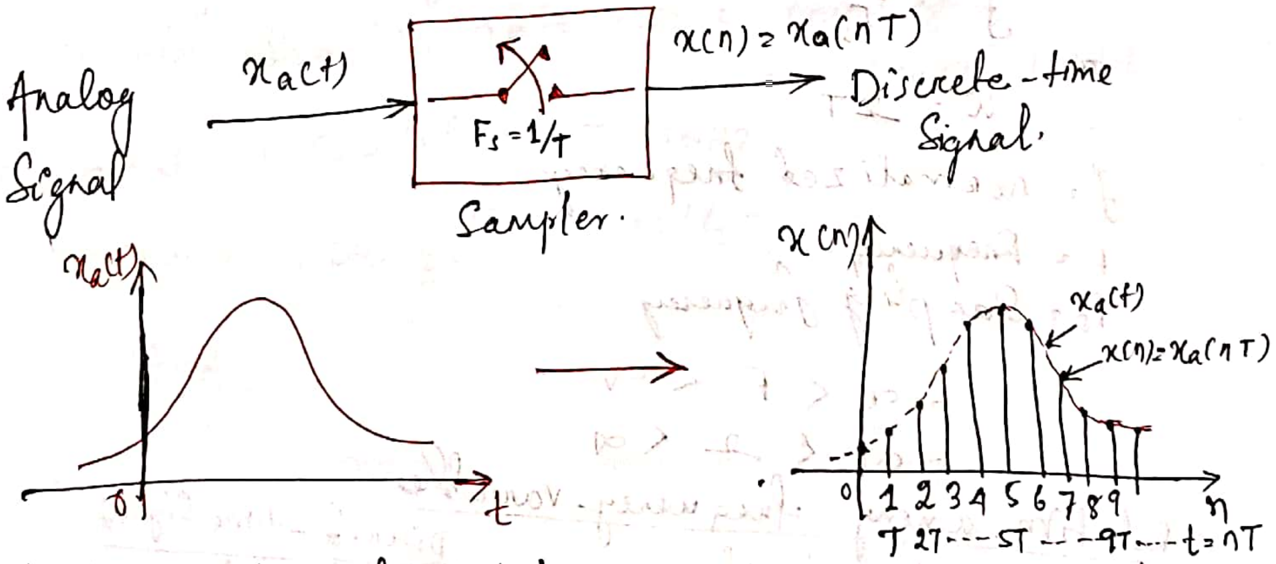
→ In principle, the analog signal can be reconstructed from the samples provided that the sampling rate is sufficiently high to avoid the problem commonly called aliasing.

1.4.1 Sampling of Analog Signals

→ There are many ways to sample an analog signal. We limit our discussion to periodic or uniform sampling,

$$x(n) = x_a(nT), \quad -\infty < n < \infty$$

Where $x(n)$ is discrete-time signal by taking of analog signal $x_a(t)$ every T seconds.



→ The time interval T between successive samples is called the sampling period or sample interval & it's reciprocal $1/T = F$, is called the sampling rate (samples per second) or the sampling frequency (Hz).

→ Periodic Sampling establishes a relationship betⁿ the time variables t and n of continuous-time and discrete-time signals respectively. these variable are linearly related through the Sampling period T or, equivalently through the Sampling rate $F_s = 1/T$

$$t = nT = \frac{n}{F_s}$$

there exists a relationship betn the frequency variable F (or Ω) for analog Signals & the frequency variable f (or ω) for discrete-time signals.

$$x_a(t) = A\cos(2\pi Ft+\theta) \quad -①$$

which when sampled periodically at a rate $F_s = 1/T$ samples per Second,

$$x_a(nT) \equiv x(n) \equiv A\cos(2\pi FnT+\theta)$$

$$= A\cos\left(\frac{2\pi nF}{F_s}+\theta\right) \quad - ②$$

If we compare 2 & $x(n) = A\cos(2\pi fn+\theta)$, $-\infty < n < \infty$

$$f = \frac{F}{F_s}$$

or

$$\omega = \Omega T$$

f = normalized frequency

F = Frequency.

F_s = Sampling frequency

$$-\infty < F < \infty$$

$$-\infty < \Omega < \infty$$

Relation among frequency variable

Continuous-time Signal

$\Omega = 2\pi F$

$\frac{\text{radians}}{\text{sec}}$ Hz

$\omega = \Omega T$, $f = F/F_s$ ⟶

Discrete-time Signal

$\omega = 2\pi f$

$\frac{\text{radians}}{\text{sample}}$, $\frac{\text{cycle}}{\text{sample}}$

$-\pi \leq \omega \leq \pi$

$-1/2 \leq f \leq 1/2$

$-\infty < \Omega < \infty$ $\xleftarrow{}$ $\Omega = \omega/T$, $F = f \cdot F_s$ $-\pi/T \le \Omega \le \pi/T$

$-\infty < F < \infty$ $-F_s/2 \le F \le F_s/2$

Discrete -time Signal is $\omega = \pi$ or $f = \frac{1}{2}$ with a Sampling rate F_s the corresponding highest values of F & Ω

$$\boxed{\begin{aligned} F_{max} &= \frac{F_s}{2} = \frac{1}{2T} \\ \Omega_{max} &= \pi F_s = \frac{\pi}{T} \end{aligned}}$$

Ex analog Signal

$x_a(t) = 3 \cos 100\pi t$

a) Determine the minm Sampling rate required to avoid aliasing

b) Suppose that the Signal is Sampled at the rate $F_s = 200Hz$ what is the ~~dee~~ discrete -time Signal obtained after Sampling?

Ans

a) analog frequency $F = 50Hz$.

minm Sampling rate freqn $=$ $F_{min} = \frac{F_s}{2}$

$= F_s = 2 \times 50 = 100 Hz$

b) Signal is Sampled at $F_s = 200Hz$

$x(n) = 3 \cos \frac{100\pi}{200} = 3 \cos \pi/2 \, n$

1.4.2 The Sampling Theorem

~~Let $x_a(t) = \sum_{i=1}^{N} A_i \cos(2\pi F_i t + \theta_i)$~~

A continuous time Signal can be completely represented in it's samples & recovered back, if the sampling frequency f_s is greater than or equal to the twice of the highest frequency component of the message signal (f_m)

Fmax = 3000 Hz for the class of Speech Signals & Fmax = 5 MHz for television. Signal.

from our knowledge of Fmax we can select the appropriate Sampling rate $F_s = 1/T$ is $F_s/2$.

any frequency above $F_s/2$ or below $-F_s/2$ results in Samples that are identical with a corresponding. Any frequency range $\boxed{-F_s/2 \leq F \leq F_s/2}$, to avoid corresponding aliasing. we must select the Sampling rate to be Sufficiently high. That is we must

$$\boxed{F_s/2 > F_{max}}$$

$$\boxed{F_s \geqslant 2f_{am}}$$

→ if the highest frequency contained in an analog Signal $x_a(t)$ is $F_{max} = B$ and the signal is Sampled at a rate $F_s > 2f_{max} \equiv 2B$, then $x_a(t)$ can be exactly recovered from it's Sample values using the interpolation function.

$$g(t) = \frac{\sin 2\pi Bt}{2\pi Bt}$$

Thus $x_a(t)$ may be expressed as the Sampling rate $\boxed{F_S = 2B = 2f_{max}}$ is Called Nyquist rate

1.4.3 Quantization of continuous - Amplitude Signals:-

→ As we have been, a digital is a sequence of numbers (Samples) in which each number is represented by a finite number of digits (finite precision).

→ The process of converting a discrete - time continuous amplitude signal into a digital Signal by expressing each sample value as a finite (instead of an infinite) number of digits is called quantization.

→ the error introduced in representing the continuous-valued signal by a finite set of discrete value levels is called quantization error or quantization noise.

Samples x(n) as Q[x(n)] & let $x_q(n)$ denote the sequence of quantized samples of the output of the quantizer.

$$x_q(n) = Q[x(n)]$$

the quantization error is a sequence $e_q(n)$ defined as the difference betn the quantized value & the actual sample

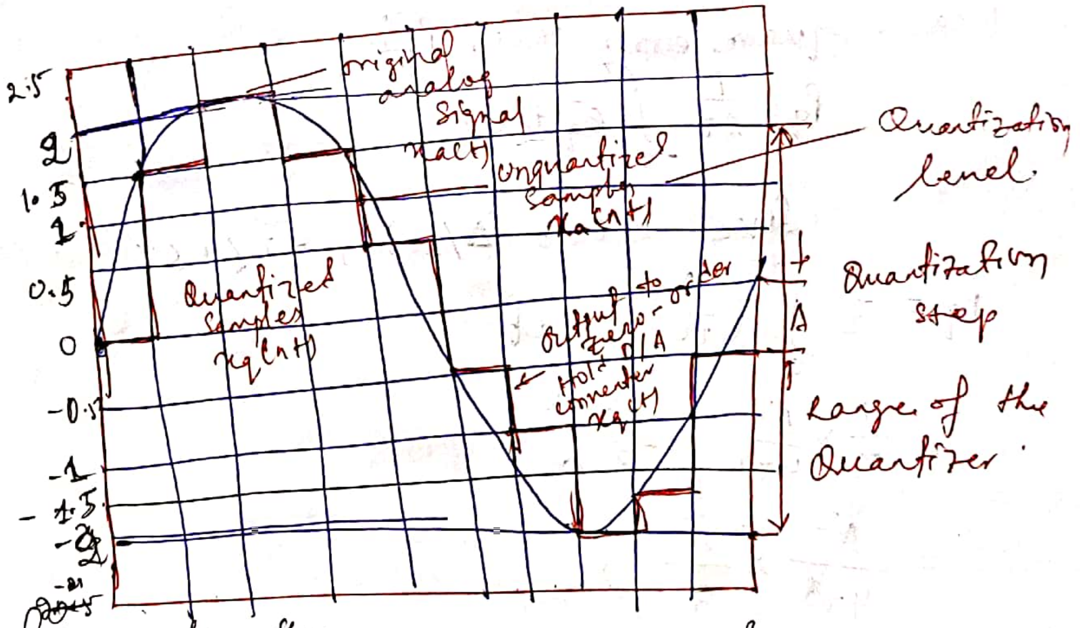
$$e_q(n) = x_q(n) - x(n)$$

$$\Delta = \frac{x_{max} - x_{min}}{L} \qquad L = 2^n$$

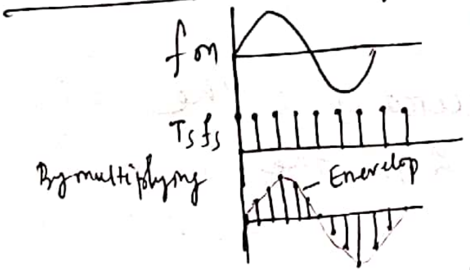
if x_{min} & x_{max} represent the minm & maxm values of x(n)

& L is the number of quantization level

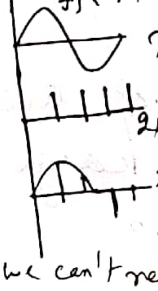
n = No. of Bits



continued from sampling theorem



$f_s < 2f_m$ message signal



$x(t) = \sin 2\pi t + \sin 3\pi t + \sin 4\pi t$

$\omega_1 = 2\pi$, $\omega_2 = 3\pi$, $\omega_3 = 4\pi$

$2\pi f_1 = 2\pi$ $\quad f_2 = 1.5 \quad f_3 = 2$

$f_1 = 1$

$f_m = max(f_1, f_2, f_3) = 2$

we can't reconstruct if $f_s < 2f_m$ because take more time period T_s↑

1.4.4 Quantization of Sinusoidal Signals

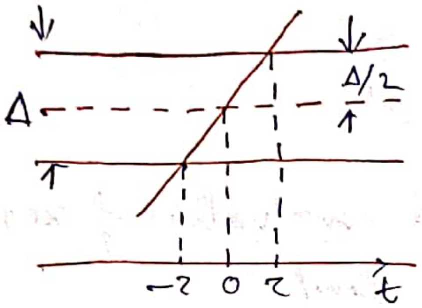
→ The Sampling & quantization of an analog Sinusoidal Signal $x_a(t) = A\cos\Omega_0 t$ using a rectangular grid.

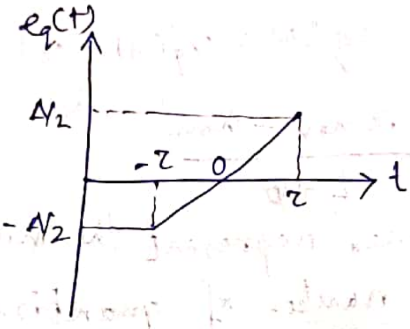
→ Horizontal lines within the range of the quantizer indicate the allowed levels of quantization.

→ vertical lines indicates the Sampling time

Thus, orginal analog Signal $x_a(t)$ we obtain a discrete-time Signal $x(n) = x_a(nT)$ by Sampling & a discrete-time discrete-amplitude Signal $x_q(nT)$ after quantization.

$$x(n) = x_a(nT)$$





The quantization error $e_q(t) = x_a(t) - x_q(t)$

mean - Square error power P_q is

$$P_q = \frac{1}{2\tau}\int_{-\tau}^{\tau} e_q^2(t)\,dt.$$

Since $e_q(t) = (\Delta/2\tau)t, \quad -\tau \le t \le \tau.$

$$P_q = \frac{1}{\tau}\int_0^\tau \left(\frac{\Delta}{2\tau}\right)^2 t^2\,dt$$

$$= \frac{1}{\tau}\times\frac{\Delta^2}{4\tau^2}\left[\frac{t^3}{3}\right]_0^\tau$$

$$= \frac{\Delta^2}{4\cancel{\tau^3}}\times\frac{\cancel{\tau^3}}{3}$$

$$= \frac{\Delta^2}{12} \quad \text{——} \; ①$$

If the quantizer has b bits of accuracy & the quantizer covers the entire range 2A, the quantization step is $\Delta = 2A/2^b$.

$$P_q = \frac{A^2/3}{2^{2b}}$$

The average power of the signal $x_a(t)$

$$P_x = \frac{1}{T_p}\int_0^{T_p} (A \cos \Omega_0 t)^2 dt = \frac{A^2}{2}.$$

→ The quality of the output of the A/D converter is usually measured by the signal to quantization noise ratio (SQNR)

$$SQNR = \frac{P_x}{P_q} = \frac{3}{2} 2^{2b}$$

Expressed in (dB)

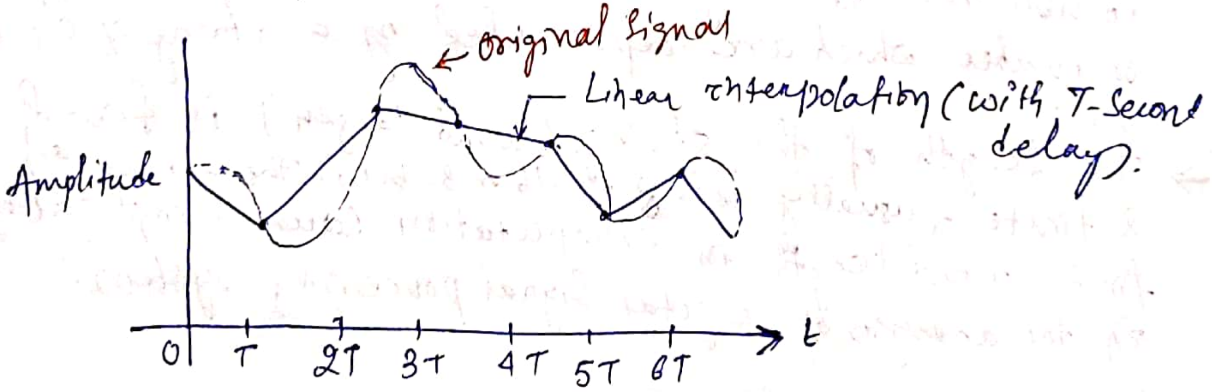
$$\boxed{SQNR\,(dB) = 10 \log_{10} SQNR = 1.76 + 6.02b}$$

1.4.5 Coding of Quantized Samples

→ The coding process in an A/D converter assigns a unique binary number to each quantization level. If we have L levels we need at least L different binary numbers.

→ With a word length of b bits we can create 2^b different binary numbers. Hence we have $2^b \geq L$, or equivalently, $b \geq \log_2 L$. Thus the number of bits required in the coder is the smallest integer greater than or equal to $\log_2 L$.

Ex It can easily be seen that we need a coder with b = 4 bits, commercially available A/D converters may be obtain with finite precision of b = 16 or less



1.4.6 Digital - to - Analog Conversion

→ To convert a digital signal in to an analog signal we can use a digital - to - analog (D/A) converter. As stated previously the task of a D/A converter is to interpolate betn samples.

→ The sampling theorem specifies the optimum interpolation for a bandlimited signal practical viewpoint the simplest D/A converter is the zero - order hold.

→ Suboptimum interpolation technique results in passing frequencies above the folding frequency. Such frequency components are undesirable and are usually removed by passing the output of the interpolator through a proper analog filter which is called a postfilter or smoothing filter.

1.4.7 Analysis of Digital Signals and Systems Versus Discrete-Time Signals & Systems.

→ we have seen that a digital signal is defined as a function of an integer independent variable & it's values are taken from a finite set of possible values.

→ The usefulness of such signal is a consequence of the possibilities offered by digital computer. computers operate on number which are represented by a string of 0's & 1's.

→ The length of this string (word length) is fixed & finite & usually is 8, 12, 16 or 32 bits. The effect of finite word length in computation cause complication in the analysis of digital signal processing system.

CHAPTER - 2

DISCRETE TIME SIGNALS & SYSTEMS

2.1 state and explain discrete time signals

→ A Discrete - time signal x(n) is a function of an independent variable that is an integer.

→ it is not defined at instants betn two successive samples

→ x(n) was obtained from sampling an analog signal $x_a(t)$, then $x(n) \equiv x_a(nT)$, where T is the sampling period (i.e time betn successive samples).

→ Some alternative representation that are often more convenient to use. These are:

All are Infinite duration

1. functional representation, Such as

$$x(n) = \begin{cases} 1, & \text{for } n = 1,3 \\ 4, & \text{for } n = 2 \\ 0, & \text{else where.} \end{cases}$$

2. Tabular representation, Such as

n	----	-2	-1	0	1	2	3	4	5	---
x(n)	----	0	0	0	1	4	1	0	0	---

3. Sequence representation

An infinite - duration signal or sequence with the time origin (n=0) indicated by the symbol ↑ is represented as

$$x(n) = \{\ldots\ldots 0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\ldots\}$$

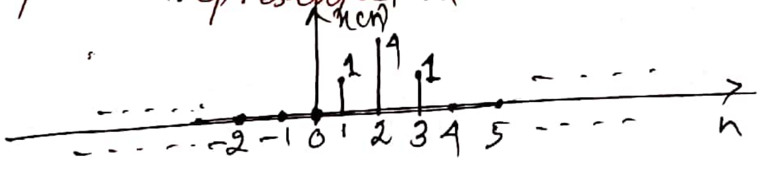
A sequence x(n), which is zero for n<0

$$x(n) = 0 \quad n < 0$$

can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, 1, 0, 0 \ldots\ldots\}$$

4. Graphical representation.



→ for Finite-duration Sequence can be represented as

$x(n) = \{3, -1, \underset{\uparrow}{-2}, 5, 0, 4, -1\}$ — Both Side Signal.

finite. duration Sequence that Satisfies the condition $x(n) = 0$ for $n < 0$

$x(n) = \{\underset{\uparrow}{-2}, 5, 0, 4, -1\}$

is called Right Sided Signal.

$x(n) = 0$ for $n > 0$

$x(n) = \{3, -1, \underset{\uparrow}{-2}\}$

is called left Side Signal.

2.1.1 Some elementary Discrete-Time Signals

1. unit Sample Sequence (Impulse Signal)

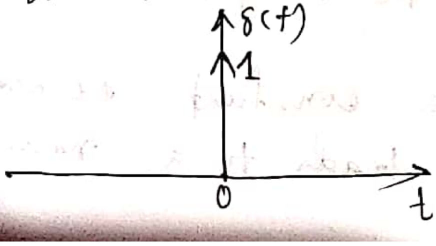
Continuous Time

→ Impulse Signal is one of the most important signal and play an important role in the analysis. unit impulse can be regarded as a rectangular pulse with a width that has become infinitly large and and overall area remain unit. Hence unit impulse signal is a signal with zero amplitude everywhere except at t=0 and t=0 the amplitude is infinite such that the area under the curve is equal to one.

✱ $$\delta(t) = \begin{cases} 0, & \text{for } t \neq 0 \\ 1, & \text{for } t = 0 \end{cases}$$

✱ $$\int_{-\infty}^{\infty} \delta(t) \cdot dt = 1$$ Area under $\delta(t) = 1$

Delayed unit impulse and impulse signal is as shown in fig



Discrete time

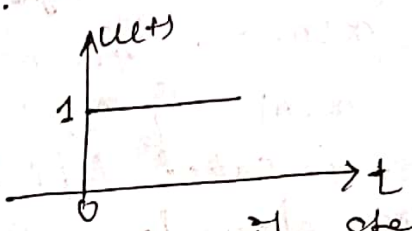
$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$$

$$\boxed{\delta[n] = u(n) - u(n-1)}$$

② Unit - Step Function

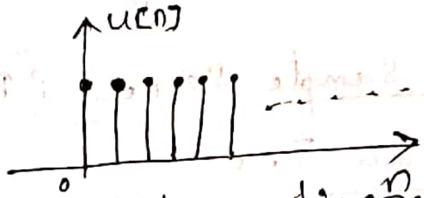
The continuous time version eqn the unit step functions is defined by .

$$u(t) = \begin{cases} 1, & t\geq 0 \\ 0, & t<0 \end{cases}$$



The Discrete time version eqn the unit step function is defined by

$$u[n] = \begin{cases} 1, & n\geq 0 \\ 0, & n<0 \end{cases}$$



A step function is said to exhibit a discontinuity at t=0 since the value of u(t) change in instantaneously from 0 to 1 at t=0.

The simple example is a DC source applied at t=0 by closing a switch.

③ Ramp Signal

The continuous time version the ramp signal or function is

$$r(t) = \begin{cases} t, & t\geq 0 \\ 0, & t<0 \end{cases}$$

$$r(t) = t \cdot u(t)$$

the impulse function δ(t) is the derivative of the step function u(t) w.r.t time. the integral of the step function u(t) is the ramp function of the unit slop.

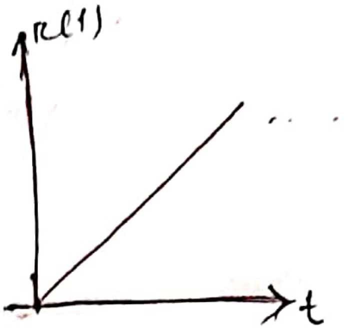
The physical example is a constant current flowing through a capacitor leads to a ramp

Voltage across capacitor.

$$V_c(t) = \frac{1}{c}\int i \cdot dt$$

$$= \frac{I}{c}\int dt$$

$$= \frac{I}{c} t.$$

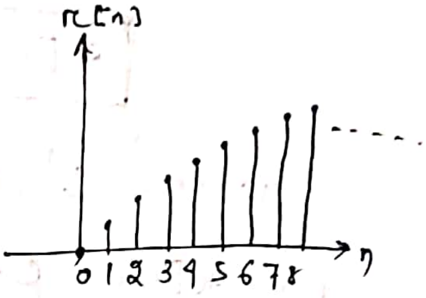


$V_c(t) \propto t$

The Discrete time version of the ramp function is defined by

$$r[n] = \begin{cases} n, & n>0 \\ 0, & n<0 \end{cases}$$

or $r[n] = n.u[n]$



④ Exponential Signals

A continuous time real exponential signal is given by,

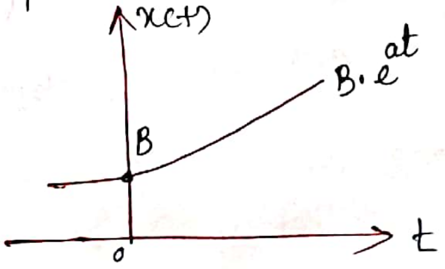
$$\boxed{x(t) = B \cdot e^{at}}$$

where B & a are real parameter.

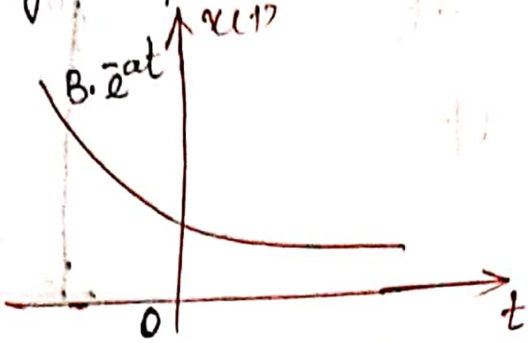
The parameter 'B' is the amplitude of the exponential signal measured at $t=0$. The behaviour of the signal is ~~defined~~ depends on the parameter 'a' and is of two types.

1) Growing exponential and 2) Decaying exponential.

* If 'a' is +ve i.e, $a>0$ then the signal $x(t)$ is called as growing exponential. This form is used in describing many physical process including chain reaction in atomic explosions and complex chemical reaction.



* if a is -ve i.e $a<0$ then the signal $x(t)$ is called Decaying exponential.



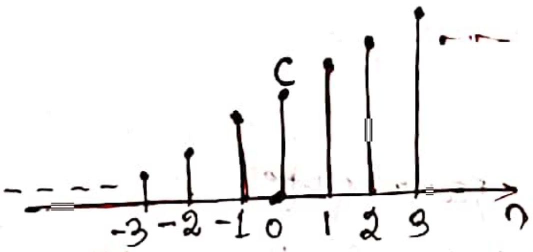
* In case of Discrete time signal

$$\boxed{x[n] = c a^n}$$

where 'c' and a are real constants.

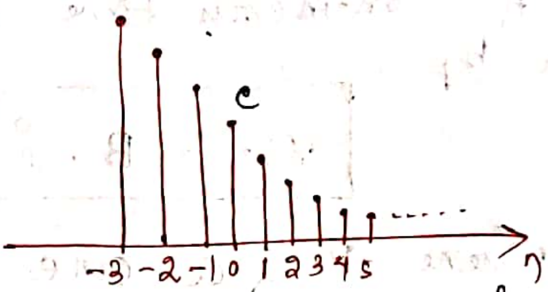
For $x[n] = c \cdot a^n$, if

1) $a>1$



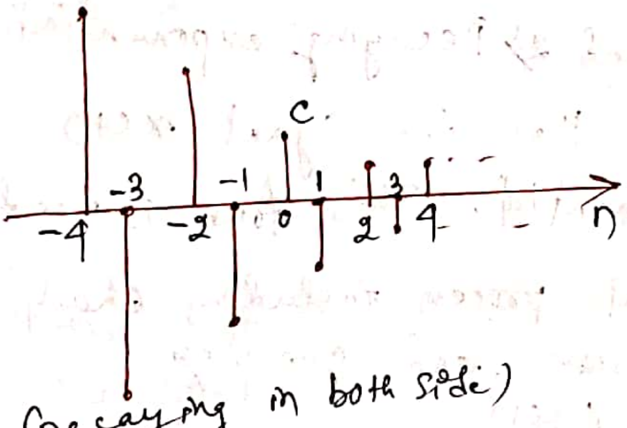
(growing signal)

2) $0 < a < 1$



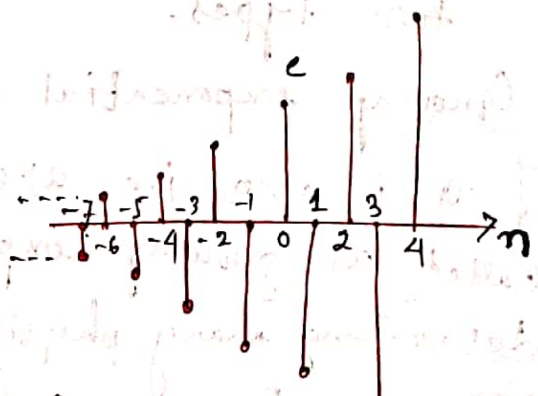
Decaying signal.

3) $-1 < a < 0$



(Decaying in both side)

4) $a < -1$



(growing in both side)

5) Sinusoidal Signals

* the contineous time version of Sinusoidal signal in it's most general from may be written as

$$x(t) = A\cos(\Omega_0 t + \phi)$$

$$x(t) = A\sin(\Omega_0 t + \phi)$$

A = amplitude , ω → Angular frequency in radian/sec

φ = phase angle in radian.

∴ A Sinusoidal signal is example of periodic Signal with $\boxed{T = \frac{2\pi}{\Omega_0}}$ i.e $\boxed{x(t) = x(t+T)}$

Angular frequency $\Omega = \frac{2\pi}{T}$ radian/sec

Proof: $x(t+T) = A\cos[\Omega(t+T) + \phi]$

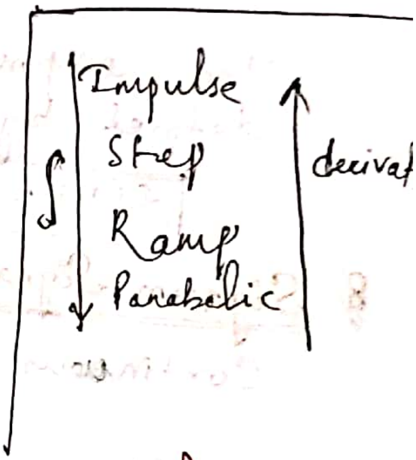
$= A\cos(\Omega t + \Omega T + \phi)$

$= A\cos(\Omega t + \Omega\frac{2\pi}{\Omega} + \phi)$

$= A\cos(\Omega t + 2\pi + \phi)$

$= A\cos(\Omega t + \phi)$

$x(t+T) = x(t)$ Proved,



* Discrete time version of a Sinusoidal Signal

The period of periodic discrete time Signal is measured in Samples - Thus x(n) is Said to be periodic with a period of N Sample.

$$x[n] = A\cos[\omega n + \phi]$$

for periodic ⇒ $x(n+N) = x[n]$

∴ $x(n) = A\cos(\omega(n+N) + \phi)$

only if $\omega N = 2\pi m$ radians

i.e $\boxed{N = \frac{2\pi m}{\omega}}$ Sample, where m, N are integers

As in contineous time signals, this is not periodic for any arbitary value of Ω. For the discrete time signals to be periodic. the angular frequency must be integer multiple of 2π.

6) Complex Exponential Signal:-

The complex exponential signal is defined by

$$\boxed{x(t) = A e^{j\Omega_0 t}}$$

$A \rightarrow$ amplitude

$\Omega_0 =$ angular frequency

$\Omega_0 = 2\pi f_0 = \frac{2\pi}{T}$

complex exponential function can be resolved in to real and imaginary part i.e

$$x(t) = A \cdot e^{j\Omega_0 t}$$

$$= A\left[\cos \Omega_0 t + j \sin \Omega_0 t\right]$$

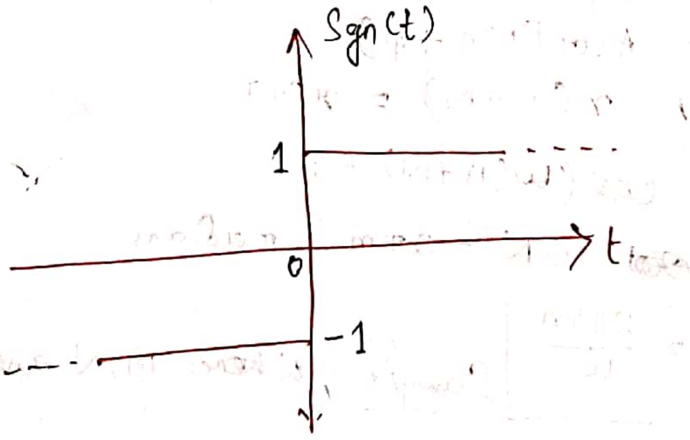
The expontial Discrete time signal is defined by

$$\boxed{e(n) = a^n \cdot x(n)}$$

8) Signum Signal

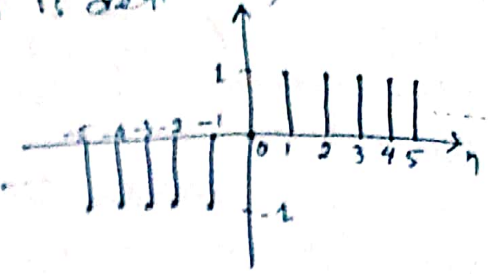
Continuous time Signum signal is defined as

$$\mathrm{Sgn}(t) = \begin{cases} 1 & : t > 0 \\ 0 & : t = 0 \\ -1 & : t < 0 \end{cases}$$



Discrete-time Signum Signal is defined as

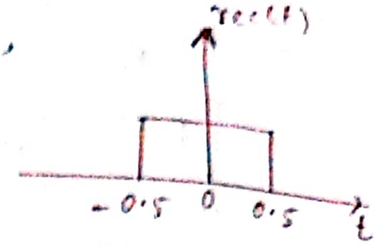
$$\mathrm{Sgn}(n) = \begin{cases} 1 : & n > 0 \\ 0 : & n = 0 \\ -1 : & n < 0 \end{cases}$$



⑨ Rectangular function

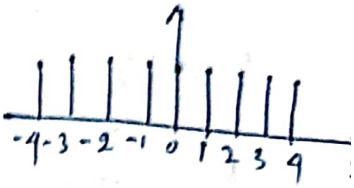
The rectangular function is defined as,

$$\mathrm{rec}(t) \text{ or } \pi(t) = \begin{cases} 1 : & |t| < 1/2 \\ 1/2 : & |t| = 1/2 \\ 0 : & |t| > 1/2 \end{cases}$$



Discrete rectangular function is

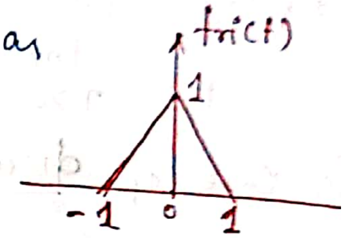
$$\mathrm{rect}\ N_0(n) / \pi[n] = \begin{cases} 1 : & |n| < 4 \\ 1/2 : & |n| = 4 \\ 0 : & |n| > 4 \end{cases}$$



⑩ Unit triangular function

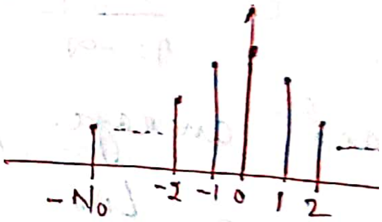
The triangular function is defined as

$$\mathrm{tri}(t) = \begin{cases} 1 - |t| & : |t| \leq 1 \\ 0 & : |t| > 1 \end{cases}$$



The discrete time triangular function is defined as

$$\mathrm{tri}(n) = \begin{cases} 1 - \frac{|n|}{N_0} & : |n| \leq N_0 \\ 0 & : |n| \geq N_0 \end{cases}$$



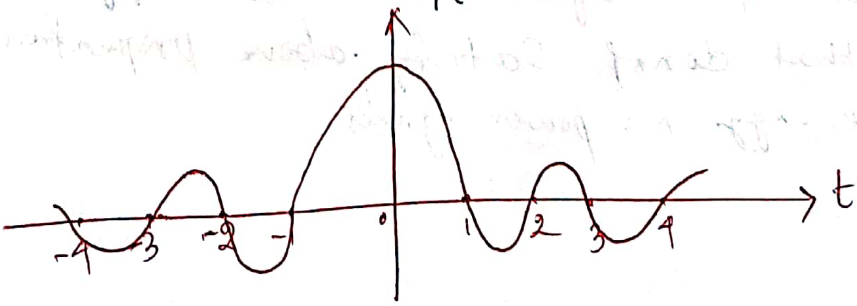
⑪ Sinc function

The Sinc function is defined as

$$\mathrm{Sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

At $t = 0$:

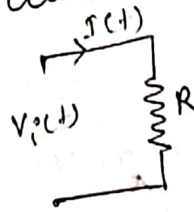
$$\mathrm{Sinc}(t)\Big|_{t=0} = 1$$



2.1.2 Classification of Discrete - Time Signa

In CT Energy & Power Signal

→ In electrical sim, Signal may represents voltage or current.



Consider voltage v(t) developed across the resistor R. producing urgent I(t). the instanteous power dissipated in ressistor R is defined by.

$$\boxed{P(t) = \frac{V^2(t)}{R}} \quad — ①$$

equivalently $P(t) = R \cdot I^2(t)$

Energy of Signal

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} x^2(t)\, dt \text{ (or) } E = \int_{-\infty}^{\infty} |x^2(t)|^2 \cdot dt = \text{Finite duration}$$

↳ (~~Non-Periodic Signal~~) Infinite duration

↳ ~~periodic~~ Signal

Average power as

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T/2}^{T/2} [x(t)]^2 \cdot dt$$

↳ (~~Non-Periodic~~) Infinite duration

or $P = \frac{1}{2T} \int_{-T/2}^{T/2} [x(t)]^2 \cdot dt$ Finite duration

In case of discrete time Signal

$$E = \sum_{n=-\infty}^{\infty} [x(n)]^2$$

and average power is

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

condition :

* A Signal is an energy Signal, if and only if the total energy of the Signal is finite for power P=0

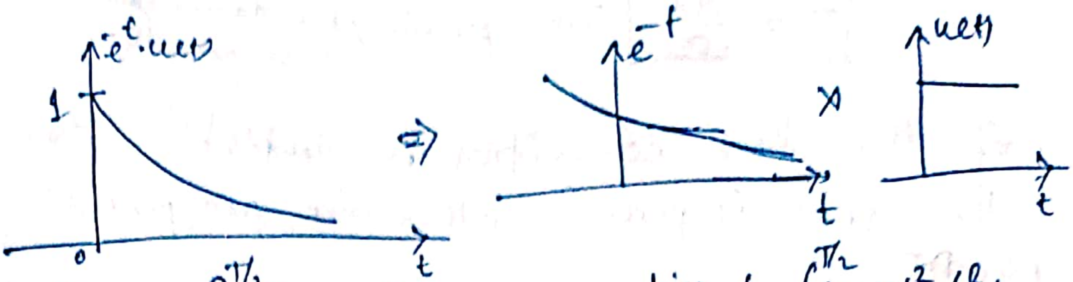
* The Signal is said to be power Signal if the average power of the signal is finite & Energy E = ∞

* The Signal that donot satisfy above properties are neither energy nor power signals.

Problem

Q Calculate Energy and power for the following C.T.S & classify it as Energy or power signal.

1. $x(t) = e^{-t} \cdot u(t)$.



$$E_{x(t)} = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t) \cdot dt$$

$$= \int_{-\infty}^{\infty} |x^2(t)|^2 \cdot dt$$

$$= \int_{0}^{\infty} (e^{-t} u(t))^2 \cdot dt$$

$$= \int_{0}^{\infty} e^{-2t} \cdot 1 \cdot dt$$

$$= \int_{0}^{\infty} e^{-2t} \, dt$$

$$= \left[\frac{e^{-2t}}{-2} \right]_0^{\infty}$$

$$= \left| \frac{e^{0} - e^{-\infty}}{-2} \right|$$

$$= 1/2$$

$$P_{av} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T/2}^{T/2} |x(t)|^2 \cdot dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T/2} e^{-2t} \cdot dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[\frac{e^{-2t}}{-2} \right]_0^{T/2}$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[-\frac{e^{-2T}}{2} + \frac{1}{2} \right]$$

$$= \frac{1}{\infty} = 0$$

(∴ This is energy signal.

* Periodic Signals and aperiodic Signals :-

For CT

$$\boxed{x(t) = x(t+T)}$$

$$\boxed{T = \frac{2\pi}{\Omega}}$$ Time periode / Fundamental Time periode

If the above condition is satisfies then the Signal is periodic otherwise non periodic.

~~For DT~~

* a contineous time Signal $x(t)$ is said to be periodic with period T, if there is a positive non-zero value of T for which . .

for all T $(-\infty \text{ to } \infty)$

* . $\boxed{x(t) \neq x(t+T)}$ is called non-periodic Signal.

For DT

A Discrete time Signal $x(n)$ is said to be periodic if it satisfies the .

$\boxed{x(n) = x(n+N)}$ for all N.

where N is a +ve integer.

N = Samples.

$\boxed{x(n) \neq x(n+N)}$ is non-periodic.

$N = 2\pi\left(\frac{m}{\omega_0}\right)$ m = Smallest integer m or min value of m.

Q. Determine whether the following signals are periodic or not if periodic find the fundamental time periode.

i) $x(t) = \cos(t + \pi/2)$

Ans - $x(t) = x(t+T)$

comparing with $x(t) = A\cos(\Omega_0 t + \phi)$

$\Omega_0 = 1$ $T = \frac{2\pi}{\Omega_0} = 2\pi$

$x(t+T) = \cos(t + 2\pi + \pi/2) \Rightarrow \cos(t + \pi/2)$

Periodic Signal.

(ii) $x(t) = 3 \sin \frac{\pi}{4} t$

$x(t) = A \sin(\Omega_0 t + \phi)$

$A = 3 \qquad \Omega_0 = \frac{\pi}{4}$

$T = \frac{2\pi}{\pi/4} = 8$ Sec.

This is periodic signals.

$x(t) = x(t+T)$

$x(t+T) = 3 \sin\left(\frac{\pi}{4}(t+T)\right)$

$= ~~3 \sin(\frac{\pi}{4}t + 8)~~$

$= 3 \sin\left(\frac{\pi}{4}t + \frac{\pi}{\cancel{4}} \times \cancel{8}^{2}\right)$

$= 3 \sin\left(\frac{\pi}{4}t + 2\pi\right)$

$= 3 \sin \frac{\pi}{4} t$

~~Thus~~ $x(t) = x(t+T)$

This periodic Signal.

(iii) $x(t) = 4 \cos\left(\frac{4\pi}{7} t + \frac{\pi}{6}\right)$

$\Omega_0 = \frac{4\pi}{7} \qquad T = \frac{\cancel{2\pi}}{\cancel{4}^{2}\cancel{\pi}/7}$

$= \frac{7}{2}$ Sec.

$x(t)$ is periodic with $\frac{7}{2}$ Sec.

(iv) $x(t) = 1 + \cos(3t + 6)$

$\Omega_0 = 3 \qquad T = \frac{2\pi}{3}$ Sec

$x(t) = x(t+T)$

$x(t+T) = 1 + \cos\left(3(t + \frac{2\pi}{3}) + 6\right)$

$= 1 + \cos(3t + 2\pi + 6)$

$= 1 + \cos(3t + 6)$

So, this is periodic Signal.

5) $x(t) = \cos^2 \pi/8\, t$

$\cos^2\theta = \frac{1}{2}[1+\cos 2\theta)$

$= \frac{1}{2}[1+\cos 2\pi/8\, t)$,

$T = \frac{2\pi}{\Omega_0}$ $\quad \Omega_0 = 2$ $\rightarrow \frac{1}{2}[(1+\cos \pi/4\, t))$

$= \frac{2\pi}{\cancel{2}\pi/4} = 8$ Sec

This is periodic Signal.

6) $x(t) = e^{j2\pi t/10}$

Given x(t) is periodic complex exponential signal comparing with

$x(t) = A e^{j\omega t}$

$\Omega_0 = \frac{2\pi}{10}$

$T = \frac{2\pi}{\Omega_0} = \frac{2\pi}{\frac{2\pi}{10}} = 10$ Sec

This periodic Signal

7) $x(t) = e^{j\omega t}$

prove x(t) is periodic

Ans for the Signal to be periodic $x(t) = x(t+T)$

So replace t by t+T

$x(t+T) = e^{j\omega(t+T)}$

$= e^{j\omega t} \cdot e^{j\omega T}$

$= e^{j\omega t} \cdot e^{j\omega \frac{2\pi}{\omega}}$

$= e^{j\omega t} \cdot e^{j\cancel{\omega}2\pi}$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta$

$e^{j2\pi} = \cos 2\pi + \overset{0}{\cancel{i\sin 2\pi}}$

$= 1$

$= e^{j\omega t} \cdot 1$

$x(t+T) = x(t)$

of is periodic

8) Determine whether the following Signals are periodic or not periodic. If periodic find the period of the Signal.

a) $x(t) = \sin \pi/3\, t + \cos \pi/4\, t$

Ans $x(t) = x_1(t) + x_2(t)$

$x_1(t) = \sin \pi/3\, t$

$x_2(t) = \cos \pi/4\, t$

$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\pi/3} = 6$ Sec

$T_1 = \frac{2\pi}{\omega_2} = 2\pi/\pi/4 = 8$ Sec

consider

$\frac{T_1}{T_2} = \frac{6}{8} = \frac{3}{4}$

rational number

x(t) is periodic.

$\frac{T_1}{T_2} = \frac{3}{4}$

$4T_1 = 3T_2 = T$

$4 \times 6 = 3 \times 8$

T = 24 Sec

9) $x(t) = \cos\sqrt{2}t + \sin t$

$x(t) = x_1(t) + x_2(t)$

$x_1(t) = \cos\sqrt{2}t$, $T_1 = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$

$x_2(t) = \sin t$, $T_2 = 2\pi$ sec.

$$\frac{T_1}{T_2} = \cancel{\frac{2\pi}{\sqrt{2}}} = \frac{\sqrt{2}\pi}{2\pi} = \frac{1}{\sqrt{2}}$$

if is rational number

x(t) is not periodic.

10) $x(n) = 5\sin 2n$

$\omega_0 = 2$

$N = \frac{2\pi}{\omega_0} m$

$= \frac{\cancel{2}\pi}{\cancel{2}}$

$= \underline{\pi} m$

$N = \pi$

x(n) is a non-periodic.

11) $x(n) = 3\sin(\pi/3\, n + \pi/6)$

$\omega_0 = \pi/3$

$N = \frac{2\pi}{\omega_0} m$

$= \frac{2\pi}{\pi/3} = 6$

N=6 Sample.

12) $x(n) = 3\cos(4\pi/3\, n + \pi/6)$

$\omega_0 = \frac{4\pi}{3}$

$N = \frac{2\pi}{\omega_0} m$

$= \frac{2\pi}{4\pi/3} m$

$= \frac{6}{4} = \frac{3}{2} m$

$N = 3$ for $\underline{m = 2}$

x(n) is periodic with a period $\underline{N = 3}$.

13 $x(n) = \cos 0.6\pi n + \sin\frac{3\pi}{8}n$

$x(n) = x_1(n) + x_2(n)$

$x_1(n) = \cos 0.6\pi n$

$N_1 = \frac{2\pi}{\omega_1} K = \frac{2\pi}{0.6\pi} K$

$= \frac{10}{3} K$

$N_1 = 10$ Sample. for $K = 3$

~~Now~~ $x_2(n) = \sin\frac{3\pi}{8}n$

$N_2 = \frac{2\pi}{\frac{3\pi}{8}} K$

$= \frac{16}{3} K$

$N_2 = 16$ for $\cancel{8}\,K = 3$

$\frac{N_1}{N_2} = \frac{10}{16} = \frac{5}{8}$

rational number

x(n) is periodic

⑭ $x(t) = \cos\pi/3\, t + \cos\pi/6\, t + \cos\pi/9\, t + 1$

Ans $x(t) = x_1(t) + x_2(t) + x_3(t)$

$x_1(t) = \cos\pi/3\, t$

$\omega_1 = \pi/3$

$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi \times 3}{\pi}$

$= 6$

$\omega_2 = \pi/6$

$T_2 = \frac{2\pi/6}{\pi} = 12$

$\omega_3 = \pi/9$

$T_3 = \frac{2\pi}{\omega_3} = \frac{2\pi \times 9}{\pi} = 18$

$\frac{T_1}{T_2} = \frac{6}{12} = 1/2$ rational

$\frac{T_1}{T_3} = \frac{6}{18} = \frac{1}{3}$ rational

x(t) is periodic

LCM (2, 3) = 6

$\frac{T_1}{T_2} = \frac{1}{2} \times \frac{3}{3} = \frac{3}{6}$

$\frac{T_1}{T_3} = \frac{1}{3} \times \frac{2}{2} = \frac{2}{6}$

$T = 6T_1$ (or) $3T_2$ (or) $2T_3$

$= 6 \times 6 = \underline{36}$

⑮ $x(n) = \cos(3\pi/4)n + \cos(4\pi/5)n + \cos\frac{5\pi}{6}n.$

$x(n) = x_1(n) + x_2(n) + x_3(n)$

$x_1(n) = \cos\left(\frac{3\pi}{4}\right)n$

$N_1 = \frac{2\pi}{\Omega_1}K$

$= \frac{\cancel{2\pi}}{3\cancel{\pi}/4}$

$= \frac{8}{3}K$

$= \frac{8}{3} \times 3$ $K = 3$

$\boxed{N_1 = 8 \text{ Samples}}$

$x_2(n) = \cos(4\pi/5)n$

$N_2 = \frac{2\pi}{\Omega_2}K$

$= \frac{\cancel{2\pi}}{\overset{}{\underset{2}{\cancel{4}}}\pi/5}$

$= \frac{5}{2}K$

$\boxed{N_2 = 5 \text{ Samples}}$

$K = 2$

$x_3(n) = \cos\left(\frac{5\pi}{6}\right)n$

$N_3 = \frac{2\cancel{\pi}}{5\cancel{\pi}/6}K = \frac{12}{5}K$

$\boxed{N_3 = 12}$ $K = 5$

Consider

$\frac{N_1}{N_2} = \frac{8}{5}$ → rational number

$\frac{N_1}{N_3} = \frac{8}{12} = \frac{2}{3}$ rational number.

$x(n)$ is a periodic Signal.

To find Fundamental time period

LCM $(5, 3) = 15$

$\frac{N_1}{N_2} = \frac{8}{5} \times \frac{3}{3} = \frac{24}{15}$

$\frac{N_1}{N_3} = \frac{2}{3} \times \frac{5}{5} = \frac{10}{15}$

$N = 15N_1 = 24N_2 = 10N_3$

$= 15(8) = 120$ Samples

Q calculate Energy & power for the following CTs and classify it as Energy or power signals.

An

i) $x(t) = 5\cos 5\omega t$

the Energy of the C.T.S is

first find out signal is periodic or aperiodic.

For $\Omega_0 = 5\omega$

$T = \frac{2\pi}{5\omega}$ not finite not periodic.

~~$x(t+T) = 5\cos(5\omega(t+\frac{2\pi}{5\omega}))$~~

~~$= 5\cos(5\omega t + 2\pi)$~~

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \cdot dt$$

$$= \lim_{T\to\infty} \int_{-T}^{T} (5\cos 5\omega t)^2 \cdot dt$$

$$= \lim_{T\to\infty} \int_{-T}^{T} 25\cos^2 5\omega t \cdot dt$$

$$= \lim_{T\to\infty} \int_{-T}^{T} 25\left[\frac{1+\cos 10\omega t}{2}\right] dt$$

$$= \lim_{T\to\infty} \frac{25}{2} \int_{-T}^{T} (1+\cos 10\omega t) \cdot dt$$

$$= \lim_{T\to\infty} \frac{25}{2} \left[t + \frac{\sin 10\omega t}{10\omega}\right]_{-T}^{T}$$

$$= \lim_{T\to\infty} \frac{25}{2} \left[T+T + \frac{\sin 10\omega T}{10\omega} + \frac{\sin 10\omega T}{10\omega}\right]$$

$= \infty$

The power P of a CTS is

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \cdot dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (5\cos 5\omega t)^2 \cdot dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 25 \cos^2 5\omega t \cdot dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 25\left(\frac{1+\cos 10\omega t}{2}\right) \cdot dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \cancel{\int_{-T}^{T}} \frac{25}{2} \int_{-T}^{T} (1+\cos 10\omega t)\, dt$$

$$= \lim_{T\to\infty} \frac{25}{4T} \left[t + \frac{~~\sin~~ \sin 10\omega t}{10\omega}\right]_{-T}^{T}$$

$$= \lim_{T\to\infty} \frac{25}{4T} \left[T + T + \cancel{\frac{\sin 10\omega T}{10\omega} + \frac{\sin 10\omega T}{10\omega}}\right]$$

$$= \lim_{T\to\infty} \frac{25}{4T} \left[2T + \cancel{2\frac{\sin 10\omega T}{10\omega}}^{\,0}\right]$$

$$= \lim_{T\to\infty} \frac{25}{\cancel{4T}_{2}} [\cancel{2T})$$

$$= \underline{\underline{12.5\,W}}$$

Short cut

For sinusoidal signal is always a power signal.

$x(t) = A\cos(\omega t + \theta)$

$= A\sin(\omega t + \theta)$

$$\boxed{P = \frac{A^2}{2}}$$

Q: $x(t) = t\,u(t)$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt$$

$$= \int_{0}^{\infty} t^2 dt$$

$$= \left[\frac{t^3}{3}\right]_0^{\infty} = \infty$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |t\,u(t)|^2 \cdot dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} (t^2)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \left[\frac{t^3}{3}\right]_0^{T/2}$$

$$= \lim_{T\to\infty} \frac{1}{T} \times \frac{T^3}{24} = \infty$$

Niether power Nor energy signal.

Q. $x(t) = e^{-5t} . u(t)$
find Energy & power.

So $T = \frac{2\pi}{-5}$. irrational number non periodic

$$E = \lim_{T\to\infty} \int_{-\infty}^{\infty} |x(t)|^2 . dt$$

$$= \lim_{T\to\infty} \int_{-T}^{T} |e^{-5t} u(t)|^2$$

$$= \lim_{T\to\infty} \int_{0}^{T} e^{-10t} . dt$$

$$= \lim_{T\to\infty} \left[\frac{e^{-10t}}{-10}\right]_0^T$$

$$= \lim_{T\to\infty} \left[\frac{e^{-10T}}{-10} + \frac{1}{10}\right]$$

$$= \lim_{T\to\infty} \left[\frac{1}{10} - \frac{e^{-10T}}{10}\right]$$

$$= \frac{1}{10} - \frac{0}{10} = \frac{1}{10} J$$

This is energy Signal.

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |e^{-5t} . u(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} e^{-10t} . dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[\frac{e^{-10t}}{-10}\right]_0^T$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[\frac{e^{-10T}}{-10} + \frac{1}{10}\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[\frac{1}{10} - \frac{0}{10}\right]$$

$$= 0 \ \omega$$

Q. $x[n] = a^n . u(n)$ check whether Energy Signal or power Signal.

Ans

Case 1

for $|a| < 1$

Ex $x(n) = (1/2)^n u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} |(1/2)^n|^2$$

formula

$$\boxed{\text{for } a<1 \quad \therefore \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}}$$

$$= \sum_{n=0}^{\infty} (1/4)^n$$

$$= 1 + \frac{1}{4} + \frac{1}{16} + \ldots\ldots \ldots + \infty$$

$$= \frac{1}{1 - 1/4} = 4/3$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} |(1/2)^n|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \left[1 + \frac{1}{4} + \frac{1}{16} \ldots\ldots \left(\frac{1}{4}\right)^N \right]$$

$$= \underline{0}$$

This energy signal when $\underline{\underline{a<1}}$.

Case-2

$a = 1$

Ex

$$x(n) = (1)^n u[n]$$

$$= u[n]$$

$$E = \sum_{n=0}^{\infty} |1|^2$$

$$= \sum_{n=0}^{\infty} 1$$

$$= \cancel{1+2+} \ldots$$

$$= 1 \cdot (\infty - 0 + 1)$$

$$= \infty$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} (1)^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1$$

$$\therefore \sum_{n=-N}^{N} b = b(N-(-N)+1)$$

$$= \underline{b(2N+1)}$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot N+1$$

$$= \underline{\underline{1/2}}$$

This is power signal.

Case-3

$a \leq -1$

$$\boxed{\because \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}}$$

$x(n) = (-2)^n u(n)$

$$E = \sum_{n=\infty}^{\infty} |(-2)^n u(n)|^2$$

$$= \sum_{n=0}^{\infty} |(-2)^n|^2$$

$$= \sum_{n=0}^{\infty} 4^n$$

$$= 1 + 4 + 16 + \cdots 4^{\infty}$$

$$= \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N-1} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N-1} |(-2)^n u(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N-1} |(-2)^n|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N-1} 4^n$$

$$\boxed{\because \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}} \quad \boxed{\text{for } a \neq 1}$$

$$\lim_{N\to\infty} \frac{1}{2N+1} \times \frac{1-4^N}{1-4} =$$

power signal

Case-4

$|a| > 1$

$x(n) = 3^n u(n)$

$$E = \sum_{n=0}^{\infty} 9^n$$

$$= \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 9^n$$

$$= \infty$$

(Niether power Nor energy signal).

Q $x(n) = e^{j(\pi/2\, n + \pi/4)}$

$$E = \sum_{n=-\infty}^{\infty} |e^{j(\pi/2\, n + \pi/4)}|^2$$

$$|e^{j(\omega t + \theta)}| = 1$$

$$= \sum_{n=-\infty}^{\infty} 1 = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |e^{j(\pi/2\, n + \pi/4)}|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \lim_{N\to\infty} \frac{2N+1}{2N+1}$$

$$= \lim_{N\to\infty} \frac{N(2+1/N)}{N(2+1/N)} = 1$$

Q. $x(n) = \sin(\pi/4\, n)$

$$E = \sum_{n=-\infty}^{\infty} |\sin^2(\pi/4\, n)|$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\because \cos 2\theta = 1 - 2\sin^2\theta$$

$$= \sum_{n=-\infty}^{\infty} \left|\frac{1-\cos(\pi/2\, n)}{2}\right|$$

$$= \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1-\cos(\pi/2\, n)}{2}$$

$$= \frac{1}{2} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \frac{1}{2} \lim_{N\to\infty} \frac{1}{2N+1} \times 2N+1$$

$$= \frac{1}{2} \lim_{N\to\infty} \frac{\cancel{N}(2+1/N)}{\cancel{N}(2+1/N)}$$

$$= \boxed{1/2}$$

Power Signal.

Q. $x(n) = u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} 1$$

$$= \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} u^2(n)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}\, N+1$$

$$= \lim_{N\to\infty} \frac{\cancel{N}(1+1/N)}{\cancel{N}(2+1/N)}$$

$$= 1/2$$

Power Signal

* Even Signal & odd Signal

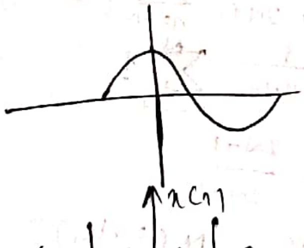
Def

A Signal $x(t)$, or $x(n)$ is said to be even Signal if it is identical to it's time reversal counter part or with reflection about the origin or symmetrical about vertical axis.

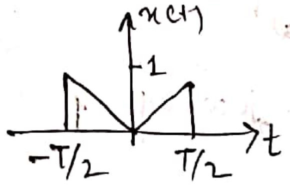
i.e $x(t)$ is even $x(-t) = x(t)$ $\forall t$

$x(n)$ is even $x(-n) = x(n)$ $\forall n$

Ex 1) $x(t) = \cos t$



②



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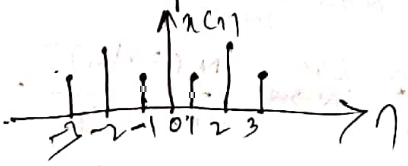
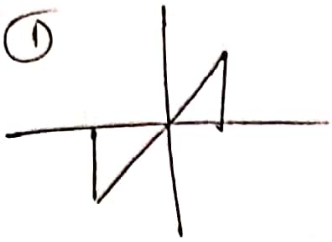


* $x(t)$ is odd

$x(-t) = -x(t)$
for all value of t

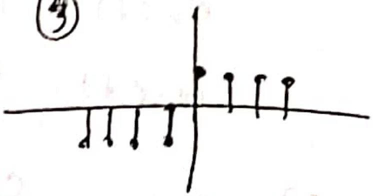
$x(-n) = -x(-n) \quad \forall n$

Eg ①



② $x(t) = \sin \omega t$

③



Decomposition of a Signal

- Any arbitrary signal $x(t)$ can be decomposed in to an even & odd signal by applying the corresponding :

$$x(t) = x_e(t) + x_o(t) \quad \text{——} ①$$

$x_e(t)$ = even component
$x_o(t)$ = odd component

$$\boxed{\begin{aligned} x_e(t) &= \frac{x(t) + x(-t)}{2} \\ x_o(t) &= \frac{x(t) - x(-t)}{2} \end{aligned}}$$

DT

$$\boxed{\begin{aligned} x_e[n] &= \frac{x[n] + x[-n]}{2} \\ x_o[n] &= \frac{x[n] - x[-n]}{2} \end{aligned}}$$

→ if the signal is complex-valued signal then a complex valued signal $x(t)$ is said to be conjugate symmetric.

if $x(-t) = x^*(t)$

Let $x(t) = \underset{\text{Real}}{a(t)} + j\underset{\text{Imaginary}}{b(t)}$

$x^*(t) = a(t) - jb(t)$

Q Find the even and odd components of the following signal.

a) $x(t) = 1+t+3t^2+5t^3+9t^4$

$x_e(t) = 1+3t^2+9t^4$ $x_o(t) = t+5t^3$

b) $x(t) = 1+t\cos t + t^2 \sin t + t^3 \sin t \cdot \cos t$

$x_e(t) = 1+t^3 \sin t \cdot \cos t$

$x_o(t) = t\cos t + t^2 \sin t$

c) $x(n) = \sin \pi/3 \, n + \cos 4\pi/3 \, n$

$x_e(n) = \cos 4\pi/3 \, n$

$x_o(n) = \sin \pi/3 \, n$

Properties

- sum of two even signal are even signal
- Sum of two odd signal are odd signal
- Sum of even signal & an odd signal is neither even or nor odd.
- product of two even signal is even signal
- product of two odd signal is ~~odd sign~~ even signal
- product of even & odd signal is odd signal.
- $\frac{\text{even signal}}{\text{odd signal}}$ = odd signal.

* Causal & Non Causal Signal

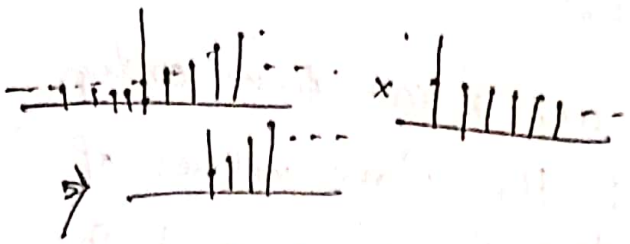
→ A signal x(n) is said to be causal if it value is zero for n<0

$$\boxed{x(n) = 0 \quad , \quad n<0}$$

otherwise the signal is non causal

Ex

$x(n) = a^n u(n)$



Causal Signal.

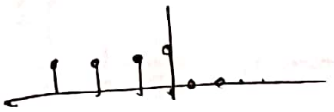
ex $x(n) = \{ \underset{\uparrow}{1}, 2, -3, -1, 2 \}$

Signal is causal.

Anti causal

A Signal is zero for all $n \geq 0$

$$\boxed{x(n) = 0 \quad \text{for all } n \geq 0}$$



anti causal sig.

2.1.3 Simple Manipulation of Discrete-Time Signals

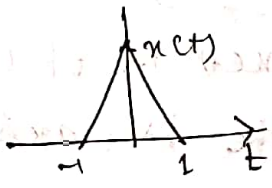
1. Time Scaling

If $x(t)$ is a contineous time signal then signal $y(t)$ is obtained by scaling the independent variable time 't' by a factor 'a' and is defind as

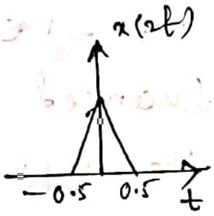
$$\boxed{y(t) = x(at)}$$

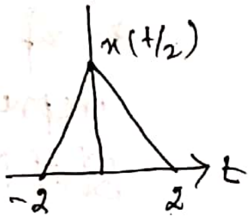
- if $a > 1$ then the resultant signal $y(t)$ is a compressed version of $x(t)$.
- if $a < 1$, then the resultant signal $y(t)$ is expanded version of $x(t)$

Ex



$x(2t)$, $x(t/2)$





Ex

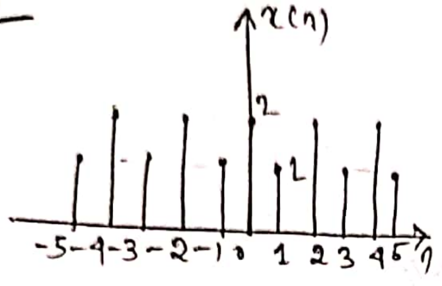
$x(2t)$, $x(t/2)$

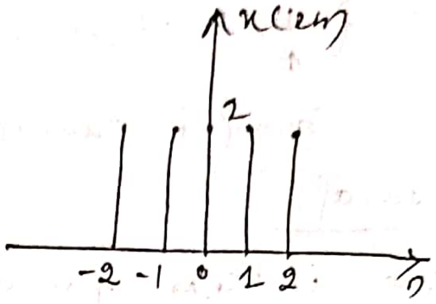
* In case of DTs

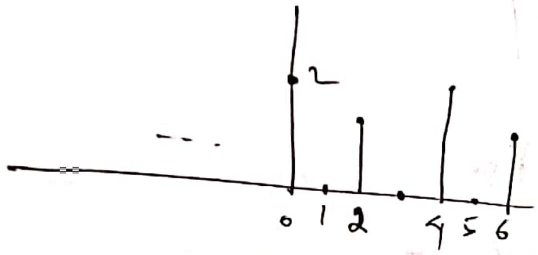
$y[n] = x[Kn]$

when $K>0$ and K can take only integer if $K>1$ then some values of the discrete time signals is lost in $y[n]$

Ex







2) Time Shifting

CT let x(t) denotes a contineous time signal then the time shifted version of x(t) is defined as

$$\boxed{y(t) = x(t \pm t_0)}$$

where t_0 = time shift.

* if $t_0 > 0$ i.e +ve then $x(t-t_0)$ represents the delayed version of the x(t) i.e it is shifted to the right relative to the time axis.

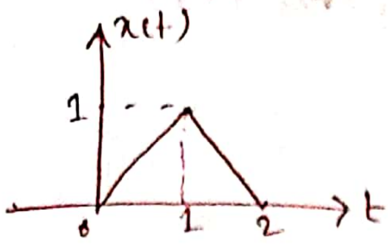
* if $t_0 < 0$ i.e -ve then $x[t-(-t_0)] = x(t+t_0)$ represents advanced version of x(t) i.e it is shifted to the left relative to the time axis.

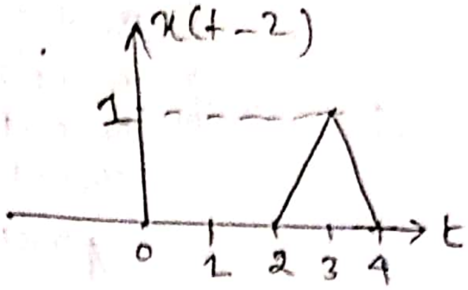
DT

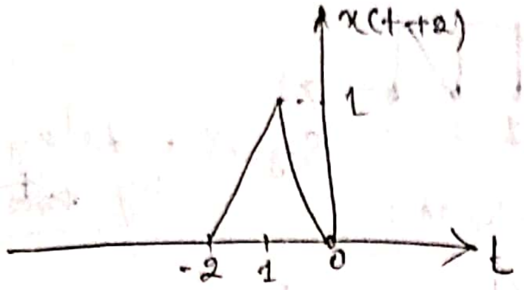
$y[n] = x[n \pm n_0]$

where n_0 is the shift either +ve or -ve integer value.

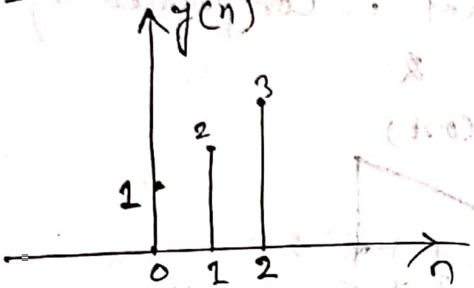
* Ex

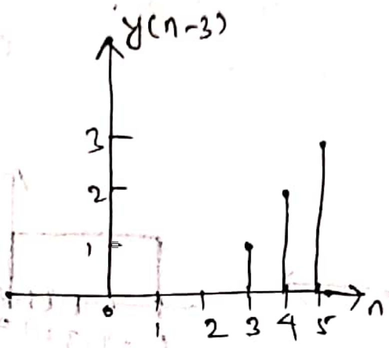


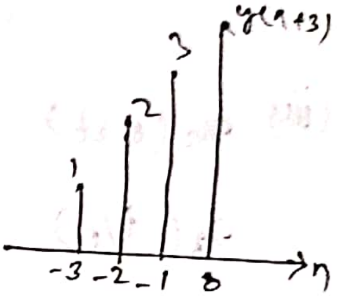




* Ex







③ Reflection or Time Reversal or Time folding

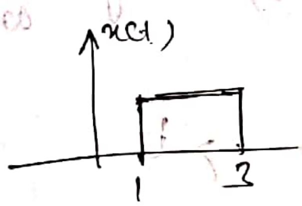
CTS ~~last y(t)~~ of x(t) is continuous time signal; then y(t) is the signal obtained by replacing time t by '-t'

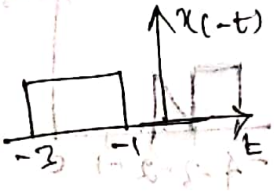
$$\boxed{y(t) = x(-t)}$$

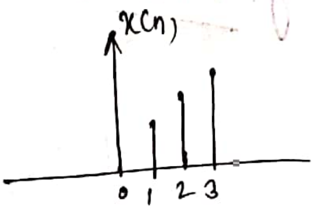
DTS

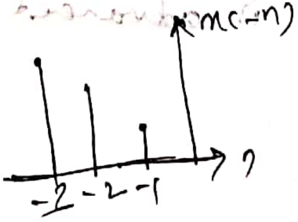
$$y(n) = x[-n)$$

Ex

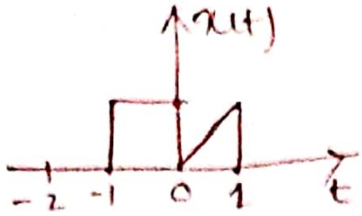








Q



find & sketch

i) $x_1(t-2)$, v) $x_5(t+3)$

ii) $x_2(2t)$

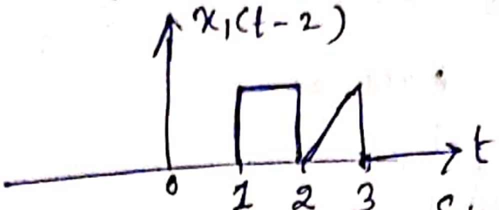
iii) $x_3(0.2t)$

iv) $x_4(-t)$

Ans

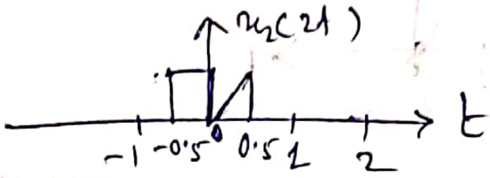
i) $x_1(t-2)$

$t = 2$ →



Signal is delay by 2

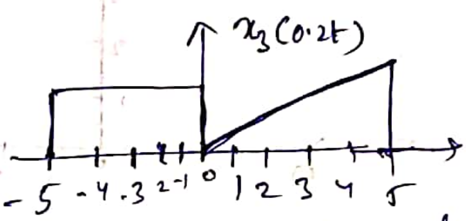
(ii) $x_2(2t)$ →



Signal is compressed by 2.

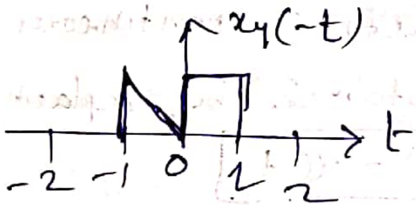
(iii) $x_3(0.2t)$

$x_3(t/5)$ →



Signal is Expand by 1/5.

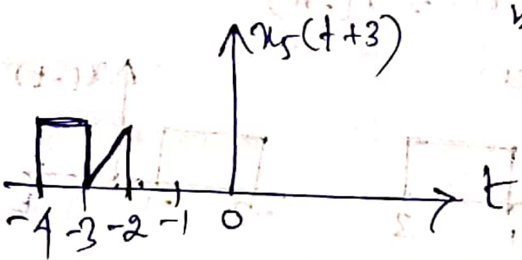
(iv) $x_4(-t)$ →



Signal is mirror image of original signal by vertical axis.

v) $x_5(t+3)$

$t = -3$ →



Signal is advanced by t+3.

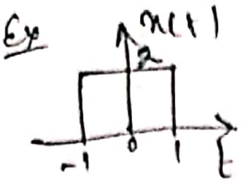
* Amplitude Scaling

e]

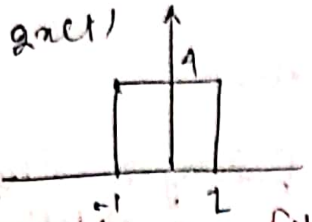
→ $x(t) = a\,x(t)$ Same as in DTS

$x[n] = a\,x[n]$

Ex



$2x(t) = ?$

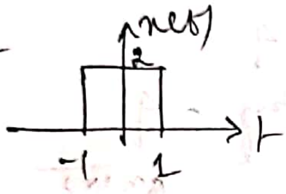


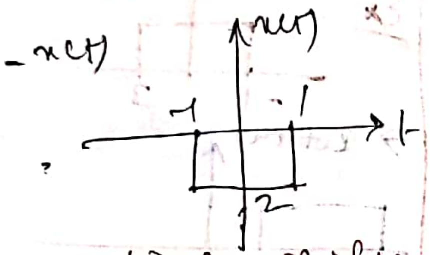
* Amplitude folding

$x(t) = -x(t)$ Same as in DTS

$x[n] = -x[n]$

Ex





* Amplitude shifting

CTS $x(t) = \pm K \pm x(t)$

DTS $x[n] = K \pm x[n]$

when

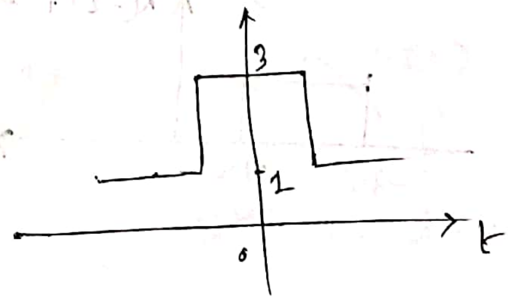
$K < 0$

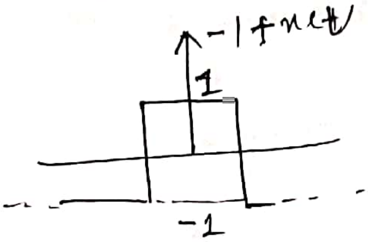
$K = -1$

Ex

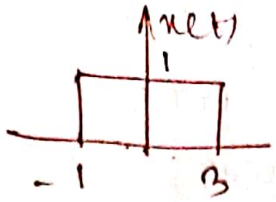
$-1 \pm x(t) = ?$

(ii) If $x(t) = ?$





Q



Draw Signal

i) $x(\cancel{2}\,2t+\cancel{8}) = ?$

ii) $x(-2t+4) = ?$

Ans

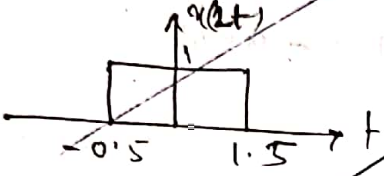
i) $x(2t+8) = ?$

$x(2(t+4)) = ?$

Step to be followed alway remember

~~Scaling → Shifting → folding~~ | Shifting → Scaling → folding

First is Scaling

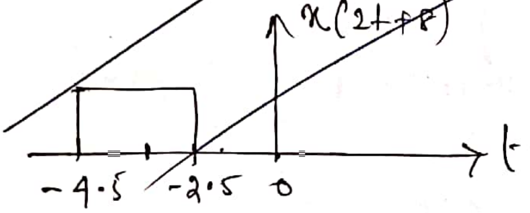


Then Shifting

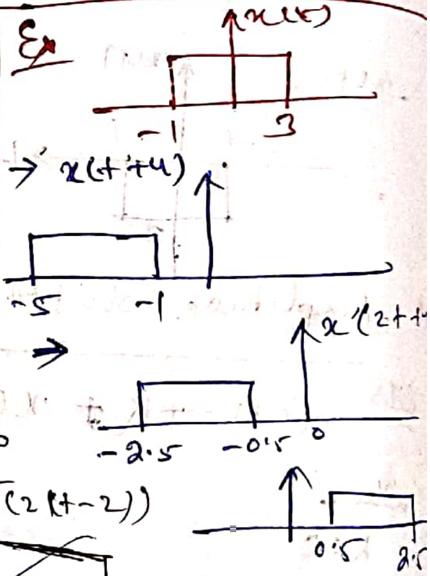
$t+4 = 0$

$t = -4$

shifting by −4.



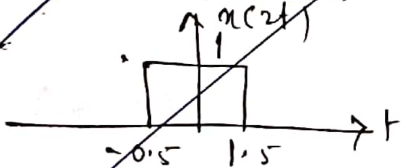
Ex



ii) $x(-2t+4) = ?$

⇒ $x(-2(t-2)) = ?$

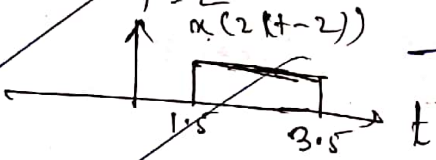
first Scaling



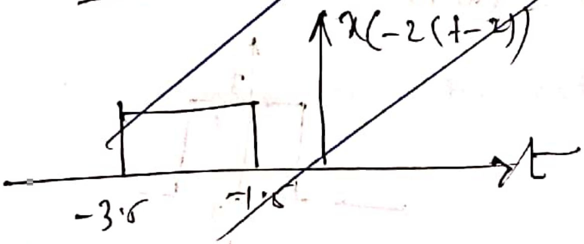
shifting

$t-2 = 0$

$t = 2$



folding



⑤ Addition of Signal

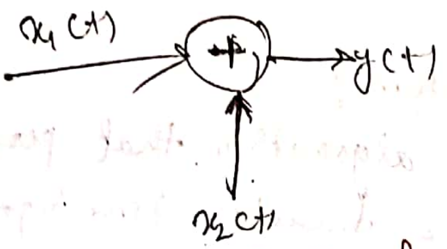
→ if $x_1(t)$ and $x_2(t)$ are the two CTS then

$$\boxed{y(t) = x_1(t) + x_2(t)}$$

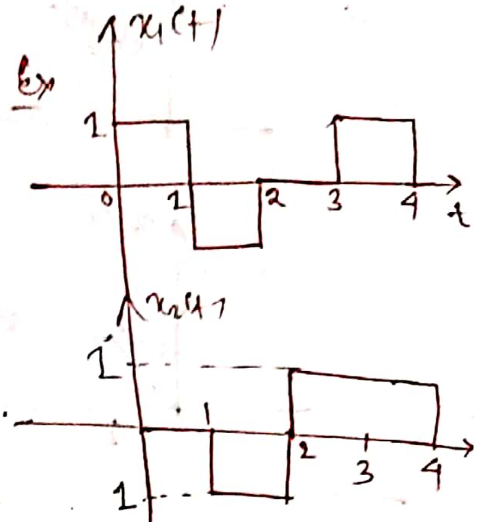
In case of DTS

$$\boxed{y(n) = x_1(n) + x_2(n)}$$

Addition is represented as

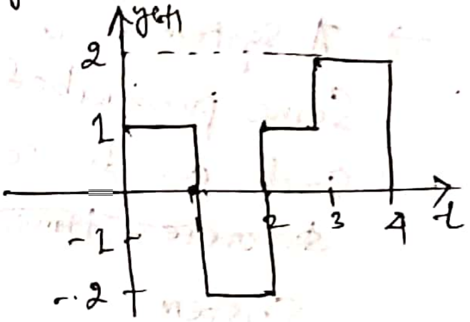


Ex



Ans

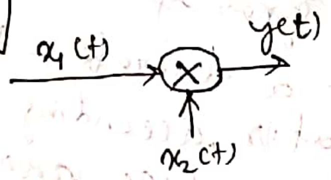
$y(t) = x_1(t) + x_2(t)$



⑥ multiplication of Signal

if $x_1(t)$ and $x_2(t)$ are the two CTS then $y(t)$ resulting for the multiplication of the signals is

$$\boxed{y(t) = x_1(t) \cdot x_2(t)}$$

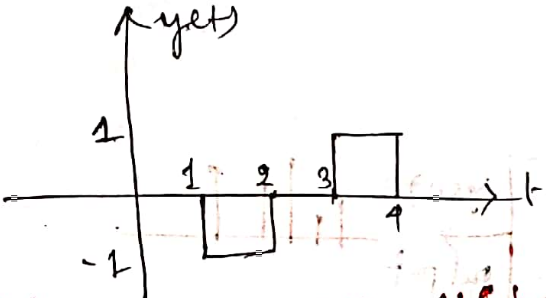


DTS

$$\boxed{y(n) = x_1(n) \cdot x_2(n)}$$

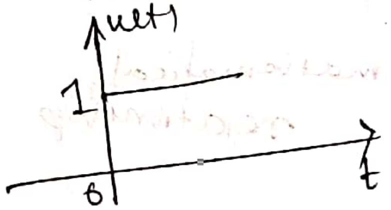
Ex we take ~~some~~ above example.

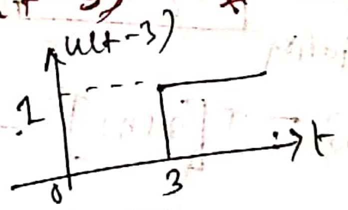
$y(t) = x_1(t) \times x_2(t)$

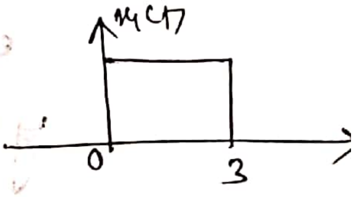


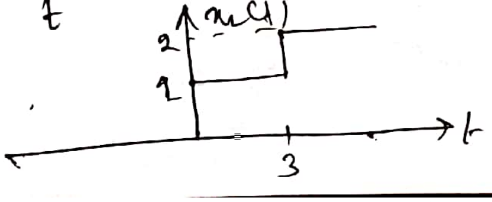
Q plot $x_1(t) = u(t) - u(t-3)$ & $x_2(t) = u(t) + u(t-3)$

Ans

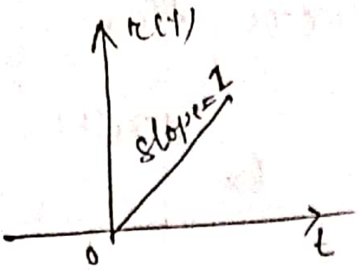


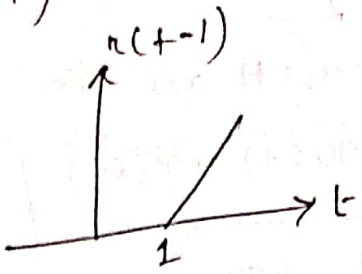


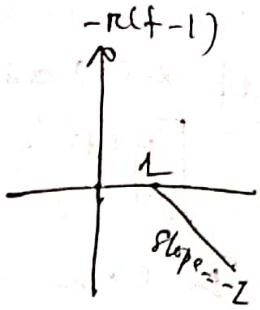


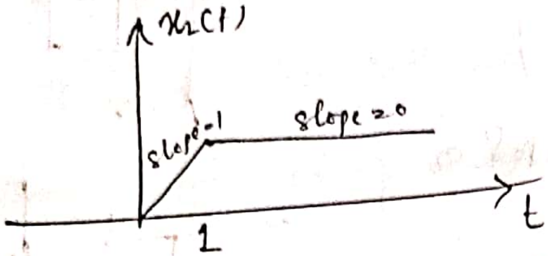


Q $x_2(t) = r(t) - r(t-1)$





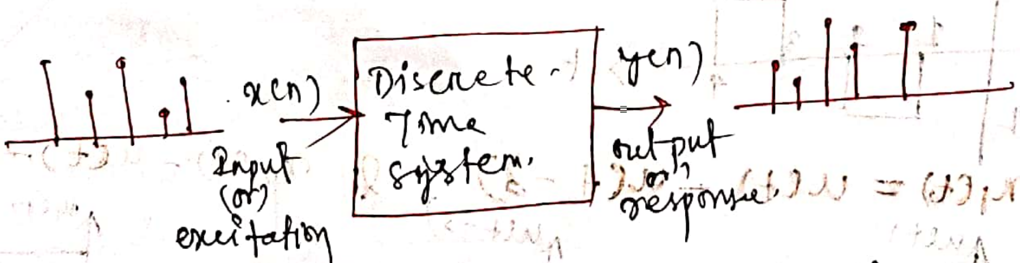




2.2 Discuss Discrete Time System :-

→ A system is a device or an algorithm that perform some prescribed operation on a discrete - time signal. ~~that operates on a~~ Such a device or algorithm ~~discrete - time signal called~~ is called Discrete - time system.

→ Discrete - time ~~signal~~ System is a device or algorithm that operates on a discrete - time signal called input or excitation, according to some well-defined rule to produce another discrete - time signal called the output or response of the system.



$y(n) \equiv T[x(n)]$ → mathematical relationship

2.2.1 Input - Output Description of System

→ The input - output description of a discrete-time system consists of a mathematical expression or a rule, defines input & output signal / relationship

$$x(n) \xrightarrow{T} y(n)$$

Ex Determine the response of the following system to the input signal.

$$x(n) = \begin{cases} |n|, & -3 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Ans

i) $y(n) = x(n)$

$x(n) = \{\ldots 0, 3, 2, 1, \underset{\uparrow}{0}, 1, 2, 3, 0, \ldots\}$

(ii) $y(n) = x(n-1)$

$\{\ldots 0, 3, 2, \underset{\uparrow}{1}, 0, 1, 2, 3, 0 \ldots\}$

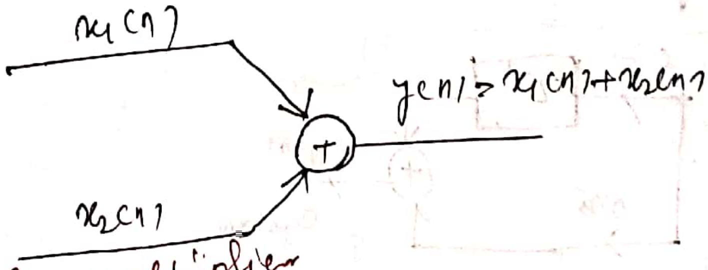
(iii) $y(n) = x(n+1)$

$\{\ldots 0, 3, 2, 1, 0, \underset{\uparrow}{1}, 2, 3, 0, \ldots\}$

2.2.2 Block Diagram representation of DTS

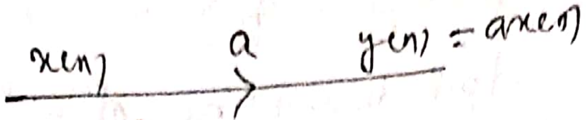
* An adder

a system adder that performs the addition of two signal sequences to form another (the sum) sequence, in other words, the addition operation is memoryless.



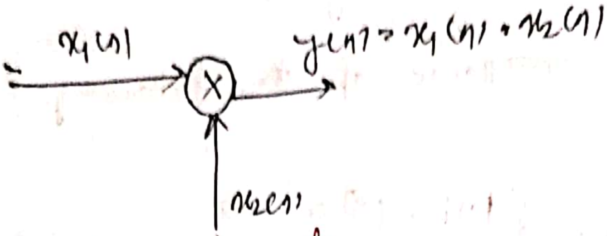
* A constant multiplier

Simple represents applying a scale factor on the input x(n)



* A signal multiplier

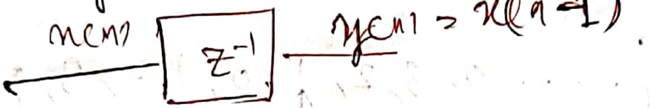
product of two signal



* A unit delay element

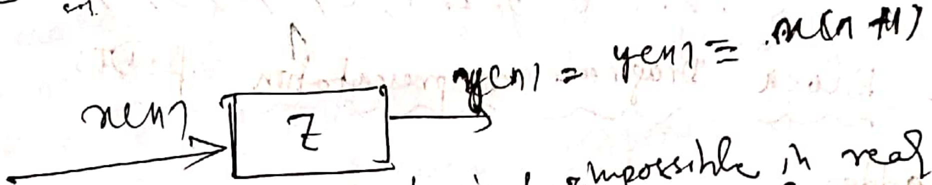
→ The unit delay is a special system that simply delays the passing through it by one sample. x(n) inputs the output x(n-1) from memory of time n to from

→ it is called $y(n) = x(n-1)$, This basic building block requires memory.



* A unit advance element

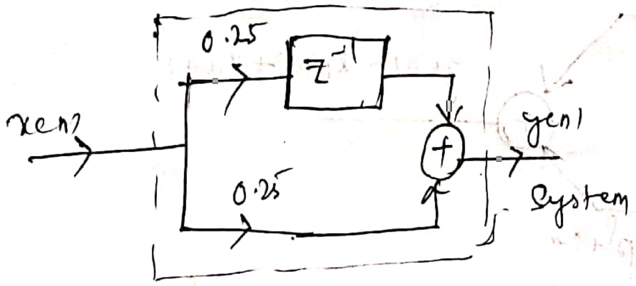
~~we can recall any sample of any~~



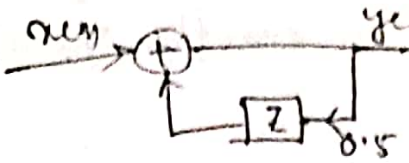
Such advance element is physical impossible in real time since infact it involved looking in to the future of the signal.

Eg. Sketch the block diagram representation of the DTS · $y(n) = \frac{1}{4}[x(n) + x(n-1)]$

An



Ex $y(n) = x(n) + \frac{1}{2} y(n+1)$



2.2.3 Classify Discrete time System

① Static and Dynamic System

→ A DTS is called static or memory less if it's output depends on at any instant 'n' depends on the inputs samples at the Same time, but not on past or further samples of the input otherwise Dynamic system.

Ex (i) $y(t) = 2x(t)$
$t=0$
$y(0) = 2x(0)$
$y(1) = 2x(1)$
$y(2) = 2x(2)$
This signal is static.

(ii) $y(n) = x(n) + x(n-1) + x(n-2)$
$n=0$
$y(0) = x(0) + x(-1) + x(-2)$
-1, -2 are past value
~~So~~ This signal is Dynamic.

② Causal and Non Causal System

→ A System is Said to be Causal if the o/p of the system is independent of future values of i/p or if the o/p of the System is dependent only on the present and past value of the i/p.

→ A System is Said to be non-causal if the o/p at any instant of time depends on the future value of the i/p signal.

Ex i) $y(t) = x(t)$ → causal.
$t=0$
$y(0) = x(0)$
$y(1) = x(1)$
~~So~~ this is depends on present value is Called causal.

(ii) $y(t) = x(t + x(t-1)$
$t=0$
$y(0) = x(0) + x(-1)$
$t=1$
$y(1) = x(1) + x(0)$
This is depends past value is called Causal

(iii) $y(t) = 2x(t) + x(t+1)$
$t=0$
$y(0) = 2x(0) + x(1)$
$y(1) = 2x(1) + x(2)$
this is depends on Future value. So this Non-Causal system.

* Anti causal system

if the output of the system depends only on the future value of i/p. this is exactly opposite to causal system.

Ex. $y(t) = x(t+2)$

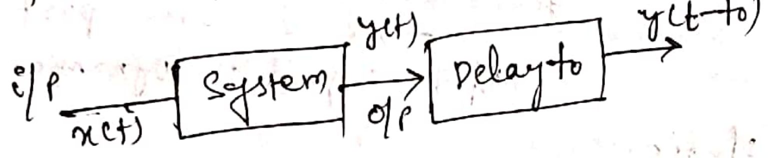
$y(0) = x(2)$

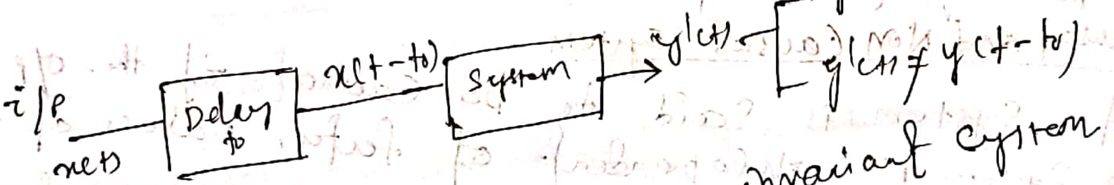
$y(1) = x(3)$

Depends on future value is called anticausal.

3) Time variant and Time invariant system

This property of systems is very important in LTI system.

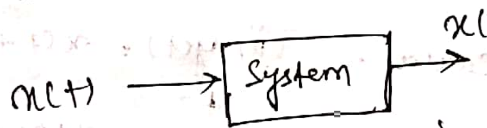




i) if $y'(t) = y(t-t_0) \Rightarrow$ Time invariant system

ii) if $y'(t) \neq y(t-t_0) \Rightarrow$ Time variant system

Ex i) $y(t) = x(2t)$



Check whether time variant or time invariant

So:

$$y(t) \xrightarrow{t_0} y(t-t_0) = x(2(t-t_0))$$
$$= x(2t-2t_0) \longrightarrow ①$$

$$x(t) \xrightarrow{t_0} x(t-t_0) \longrightarrow \boxed{\text{System}} \rightarrow x(2(t)-t_0)$$

$$y(t) = x(2t-t_0) \longrightarrow ②$$

Comparing ① & ②

$y(t) \neq y(t-t_0)$

System Time variant.

Note: Always time Scaling → Time variant.

· Amplitude Scaling → Time invariant.

Ex $y(t) = 2 + x(t)$

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = 2 + x(t)$$

Check whether time variant or Time invariant.

Sol

$$y(t) \xrightarrow{t_0} y(t-t_0) = 2 + x(t-t_0) \quad —①$$

$$x(t) \xrightarrow{t_0} x(t-t_0) \longrightarrow \boxed{S/m} \longrightarrow y(t) = 2 + x(t-t_0) \quad —②$$

By ① & ②

$y(t) = y(t-t_0)$

The given system is time invariant.

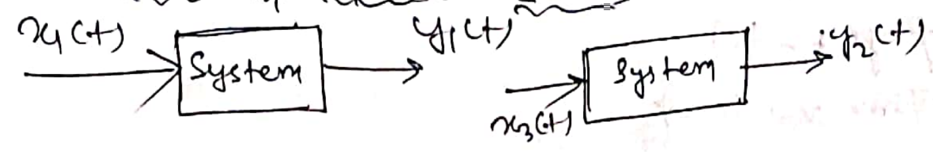
4) Linearity or Non-linearity

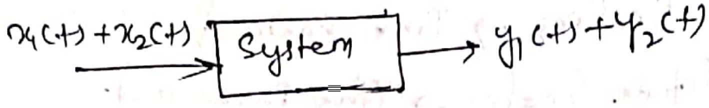
→ A system is said to be linear in terms of the system input excitation $x(t)$ and the system output response $y(t)$ if it satisfies the following propert i.e. Law of Superposition and law of homogenity

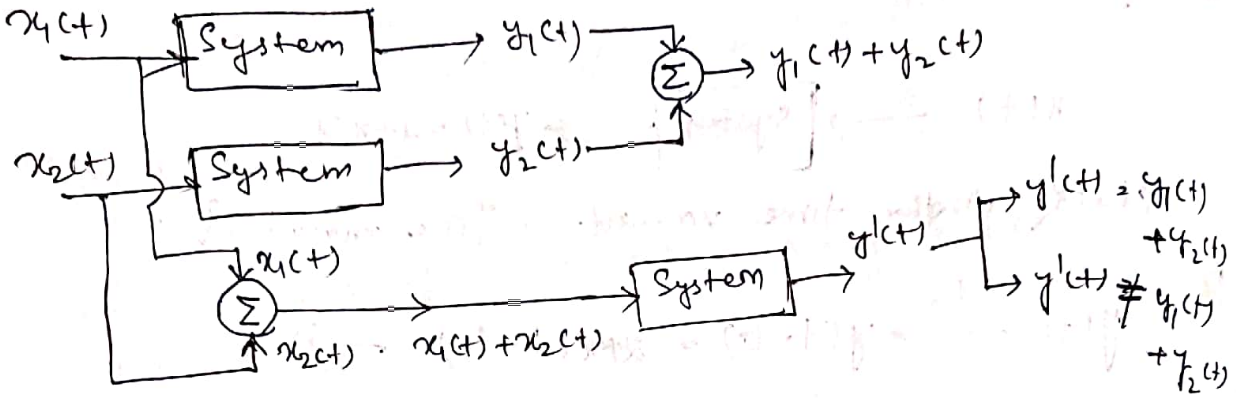
1) Law of Superposition is also called as law of additivity [LDA)
2) Law of Homogenity is also called as law of multiplication ~~or~~ (LOM) or Scalar multiplication.

→ A system which does not satisfies any of the above proper then it is called as non-linear system

Law of Superposition [LOA]





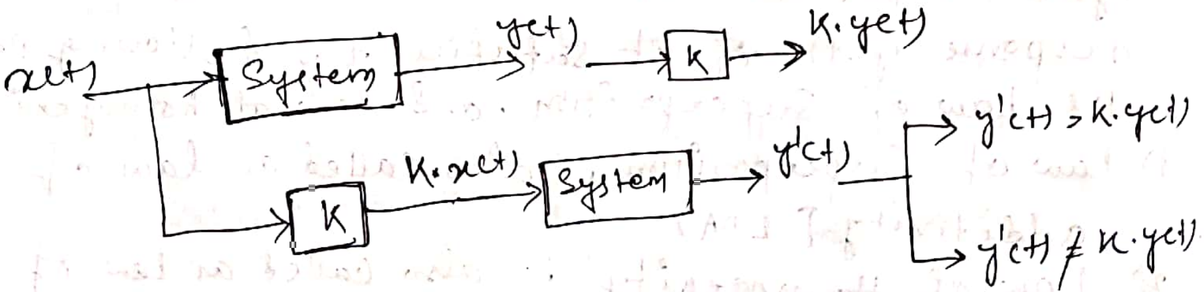


System is same in all cases

if $y'(t) = y_1(t) + y_2(t)$ ⇒ system is following law of additivity.

$y'(t) \neq y(t) + y_2(t)$ ⇒ system is not following Law of addivity.

Law of Homogenity [LOH]



if $y'(t) = k\cdot y(t)$ ⇒ system is following law of Homogenity.

$y'(t) \neq k\cdot y(t)$ ⇒ System is not following Law of Homogenity.

Ex $y(t) = x(\sin t)$

1) Law of additivity

$$x(t) \longrightarrow \boxed{System} \longrightarrow y(t) = x(\sin t)$$

$y_1(t) = x_1(\sin t)$, $y_2(t) = x_2(\sin t)$

$y_1(t) + y_2(t) = x_1(\sin t) + x_2(\sin t)$ → ①

$$x_1(t) + x_2(t) \longrightarrow \boxed{System} \longrightarrow y'(t) = x_1(\sin t) + x_2(\sin t). \quad — ②$$

① = ② ⇒ LOA is followed.

2) LOH

$y(t) = x(\sin t)$

$K \cdot y(t) = K \cdot x(\sin t)$ — ①

$$K \cdot x(t) \longrightarrow \boxed{System} \longrightarrow y'(t) = K \cdot x(\sin t) \quad — ②$$

① = ② ⇒ LOH is followed.

Both law of Superposition and law of Homogeneity are followed. Hence the given System is linear System.

⑤ Stability

A System is Said to be bounded input, bounded o/p [BIBO] stable if and only if every bounded input results in bounded o/p. The output of Such a System does not diverge if the input does not diverge.

i.e for input $|x(t)| \leq M_x < \infty$ for all.

for o/p $|y(t)| \leq M_y < \infty$ for all.

M_x & M_y represents to finit positive number

Ex – For bounded Signals are dc value, sin t, cos t, u(t) constant. – -1 to 1, 1 to -1, 0 or 1.

Problem

* Find whether the following Signals are Static or Dynamic

1) $y(n) = \log.[x(n)]$.

$n=0$

$y(0) = \log[x(0)]$

$n=1$

$y(1) = \log[x(1)]$

$n=2$

$y(2) = \log[x(2)]$

System is static.

2) $y(n) = x(n).x(n-1)$;

$n=0$

$y(0) = x(0).x(-1)$

$n=1$

$y(1) = x(1).x(0)$

$n=2$

$y(-1) = x(-1)x(-2)$

it shows past value this System is Dynamic

3) $y(n) = x^2(n) + x(n)$: Static

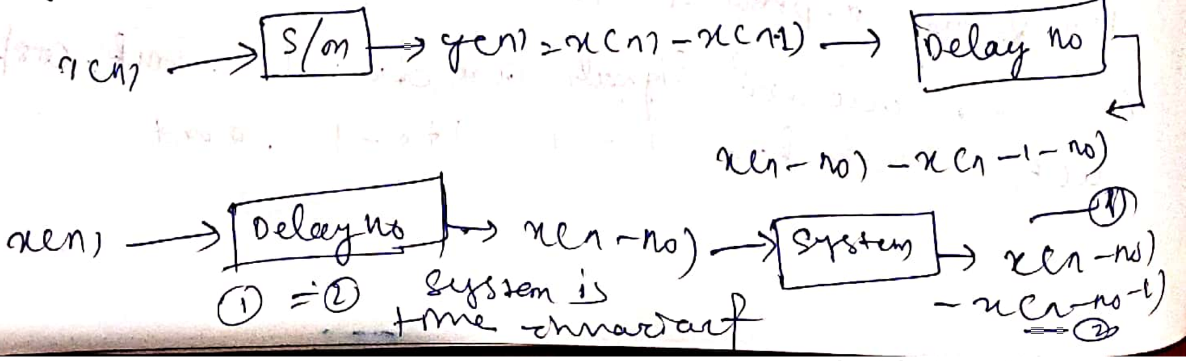
4) $y(n) = A\,x(n)$: Static

5) $y(n) = \sum_{k=0}^{\infty} x(n-k)$: Dynamic

6) $y(n) = e^{x^2(n)}$: Static

* Test whether the following Systems are time invariant:

1) $y(n) = x(n) - x(n-1)$



① = ② System is time invariant

② $y(n) = n x(n)$

$$x(n) \rightarrow \boxed{System} \rightarrow y(n) \rightarrow \boxed{Delay\ n_0} \rightarrow y(n-n_0)$$

$$= (n-n_0)\, x(n-n_0) \rightarrow ①$$

$$x(n) \rightarrow \boxed{Delay\ n_0} \rightarrow x(n-n_0) \rightarrow \boxed{System} \rightarrow y(n) = n\, x(n-n_0) \rightarrow ②$$

① ≠ ②

The given system is time variant.

* Determine whether the following systems are linear or Non-linear

1) $y(n) = n.x(n)$

$ax_1 \rightarrow y_1(n) = n\, a x_1(n)$

$bx_2 \rightarrow y_2(n) = bn\, x_2(n)$

$y_1(n) + y_2(n) = n\, a x_1(n) + n.bx_2(n)$

$= n\{ax_1(n) + bx_2(n)\} \rightarrow ①$

$$ax_1(n) + bx_2(n) \rightarrow \boxed{S/m} \rightarrow y(n) = n\{x(n)\}$$

$= n\{ax_1(n) + bx_2(n)\} \rightarrow ②$

① = ②

The system is linear

② $y(n) = x(n^2)$

$ax(n) \rightarrow y_1(n) = ax_1(n^2)$

$bx_2(n) \rightarrow y_2(n) = b x_2(n^2)$

$y_1(n) + y_2(n) = ax_1(n^2) + bx_2(n^2) \rightarrow ①$

$x(n) = a x_1(n) + bx_2(n)$

$$x(n) \rightarrow \boxed{S/m} \rightarrow y(n) = x(n^2) = [ax_1(n^2) + bx_2(n^2)] \rightarrow ②$$

① = ② system is ~~non~~ linear

③ $y(n) = x^2(n)$

$a_1x_1(n) \rightarrow y_1(n) = a x_1^2(n)$

$b_1x_2(n) \rightarrow y_2 = bx_2^2(n)$

$y_1(n) + y_2(n) = a x_1^2(n) + bx_2^2(n)$

$$x(n) \rightarrow \boxed{S/m} \rightarrow y(n) = x^2(n) = [a_1x_1(n) + b_1x_2(n)]^2 \rightarrow ○$$

1 ≠ 2 system is non linear.

④ $y(n) = A\,x(n) + B$

$a\,x_1(n) \longrightarrow aAx_1(n) + B$

$b\,x_2(n) \longrightarrow b\,Ax_2(n) + B$

$y_1(n) + y_2(n) = A\,a\,x_1(n) + B + A\,b\,x_2(n) + B$

$= A[a x_1(n) + b x_2(n)] + 2B \rightarrow$ ②

$x(n) = a x_1(n) + b\cdot x_2(n)$

$x(n) \longrightarrow \boxed{\text{System}} \longrightarrow y(n)$

$y(n) = A\{a x_1(n) + b x_2(n)\} + B \longrightarrow$ ②

① ≠ ② Then system is Non-lin

⑤ $y(n) = e^{x(n)}$

~~$a\,x(n) = a\,e^{x_1(n)}$~~

$a x_1(n) \rightarrow y_1(n) = a e^{x_1(n)}$

$b x_2(n) \rightarrow y_2(n) = b e^{x_2(n)}$

$y_1(n) + y_2(n) = a\,e^{x_1(n)} + b e^{x_2(n)}$ — ①

$x(n) = a x_1(n) + b x_2(n)$ —

$x(n) \longrightarrow \boxed{S/m} \longrightarrow y(n)$

$y(n) = e^{a x_1(n)} + e^{b x_2(n)}$ — ②

$1 \neq 2$

Non linear.

Determine whether the following systems are causal or Non causal

1) $y(n) = x(n) + \frac{1}{x(n-1)}$: causal.

2) $y(n) = x(n^2)$: causal.

3) $y(n) = x(n) - x(n-1)$: causal.

4) $y(n) = \sum_{K=\infty}^{\infty} x(K-n)$ = causal

⑤ $y(n) = A\,x(n)$ = causal

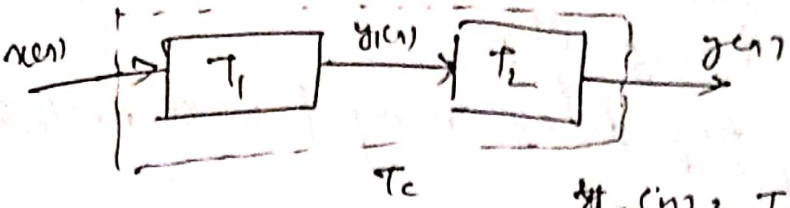
⑥ $y(n) = x(n) + 3(x+4)$ = Non causal.

⑦ $y(n) = x(2n)$. Non causal.

2.2.4 Interconnection of Discrete-Time System

Two types

cascade (series) →



$y_n(n) = T_c[x(n)]$

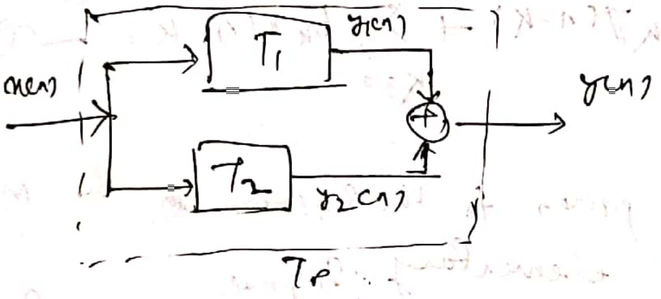
$y_1(n) = T_1[x(n)]$

$y(n) = T_2[y_1(n)]$

$= T_2[T_1[x(n)]]$

$\boxed{T_c \equiv T_1 T_2}$, But $\boxed{T_2 T_1 \neq T_1 T_2}$

Parallel



$y(n) = T_p[x(n)]$

$y_1(n) = T_1[x_1(n)]$

$y_2(n) = T_2[x_2(n)]$

$y(n) = y_1(n) + y_2(n)$

$= T_1[x_1(n) + T_2[x_2(n)]$

$T_p[x(n)] = x(n)[T_1 + T_2]$

$\boxed{T_p \equiv T_1 + T_2}$

2.3 Discrete-Time Linear Time-Invariant System

→ LTI - combination of both linear and time invariant Systems.

Linear - which satisfies both superposition & law of homogeneity.

Time invariant - Any delay in input will reflect in output.

2.3.1 Techniques for the analysis of linear system

There are two basic method linear system input-output equation

First method →

$$y(n) = F[y(n-1), y(n-2), \dots y(n-N), x(n), x(n-1) \; x(n-2) \dots x(n-M)].$$

Specifically for an LTI system.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad ①$$

Second method →

Linear system to a given to the system the input Signal in to Sum of elementary Signal.

So, that the input Signal $x(n)$ is resolved in to weighted Sum of elementary Signal components $[x_k(n)]$. So that

$$x(n) = \sum_k c_k x_k(n)$$

c_k are set of amplitudes (weighting co-efficients of the Signal $x(n)$, now elementary signal components are $x_k(n)$ is $y_k(n)$

$$y_k(n) \equiv T[x_k(n)]$$

$$y(n) = T[x(n)] = T\left[\sum_k c_k x_k(n)\right]$$

$$= \sum_{K} C_K T[x_K(n)]$$

$$= \sum_{K} C_K y_K(n)$$

2.3.2 Resolution of a Discrete-Time Signal in to impulse

~~x(n) =~~ we have an arbitary signal x(n) that we wish to resolve in to a sum of unit sample sequence. we select the elementary signal $x_k(n)$

$$x_k(n) = \delta(n-k).$$

where K represent the delay of the unit sample sequence. To handle an arbitary signal x(n).

* ~~Resolution of a D~~

$\delta(n-k)$ is zero every where except n=k where it is value unity. the result of this multiplication is another sequence that is zero every where except of n=k

$$x(n)\delta(n-k) = x(k)\delta(n-k)$$

the sequence is

$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)}$$

Q/ consider the special case of a finite-duration sequence given as

$$x(n) = \{2, 4, 0, 3\}$$

Resolve the sequence x(n) in to a sum of weighted impulse sequence.

Am

$$n = -1, 0, 1, 2$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) = \sum_{k=-1}^{2} x(k)\delta(n-k)$$

$$= x(-1)\delta(n+1) + x(0)\delta(n+0) + x(1)\delta(n-1) + x(2)\delta(n-2)$$

$$= 2\delta(n+1) + 4\delta(n) + 0.\delta(n-1) + 3\delta(n-2)$$

2.3.3 Response of LTI System to arbitary input : using convolution Sum (Theorem)

→ arbitrary input Signal $x(n)$ in to a weighted. Sum of impulse. now ~~not~~ ready to determine the response of any relaxed lihan

we denote response $y(n, K)$ of the System to the input unit sample Sequence $n = k$ by Special Symbol $h(n, k)$, $-\infty < k < \infty$

$$y(n,k) \equiv h(n,k) = T[\delta(n-k)]$$

n = time index

k = ~~po~~ location of the input impulse.

Expression of Sum weighted impulse

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

$$y(n) = T[x(n)]$$

$$= T\left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right]$$

$$y(n) = \cancel{T}\,\cancel{\sum_{k=-\infty}^{\infty}}\;\sum_{k=-\infty}^{\infty} x(k)\,T[\delta(n-k)]$$

∴ $C_k = x(k)$ = Scaled co-efficient

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)} \quad \text{—— ①} \qquad \therefore h(k) \equiv T[\delta(n)]$$

we Satisfies the Superposition property.

Now we have to Time invariant system

$$h(n) \equiv T[\delta(n)]$$

$$A(n-k) = T[\delta(n-k)$$

above eqn ①

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)}$$

* The formula gin the response y(n) of the LTI ~~sys~~ System, as input signal x(n) & the unit ~~sa~~ impulse respons h(n), is called Convolution Sum.

Suppose that we wish to compute of the System Some time instant Say $n = n_0$

$$\boxed{y(n_0) = \sum_{k=-\infty}^{\infty} x(k)\, h(n_0-k)}$$

So we have to follow four steps

1. Folding :- Fold h(k) about k=0 to obtain h(-k)
2. Shifting :- shift h(-k) by n_0 to the right ~~(left)~~ if n_0 is positive, then if ~~notn~~ shift h(-k) by ~~no of the~~ n_0 to the left, if n_0 is negative, to obtain $h(n_0-k)$.
3. multiplication :- multiply x(k) by $h(n_0-k)$ to obtain product of sequence.

$$\boxed{V_{n_0}(k) \equiv x(k)\, h(n_0-k)}$$

4. Summation :- Sum all the values of the product sequence $V_{n_0}(k)$ to ~~ot~~ obtain the value of the output at time $n = n_0$. , $-\infty < n < \infty$

Q The impulse response of a linear time-invariant System is

$$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$$

Determine the response of the System to the input signal

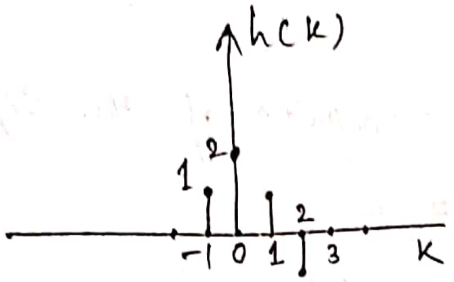
$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

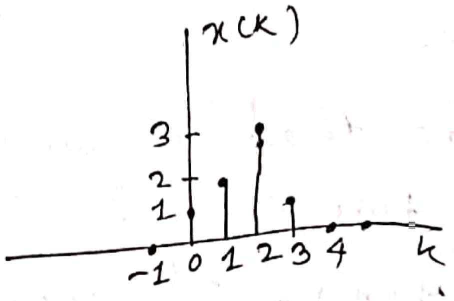
Ans

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$n = 0$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(-k)$$

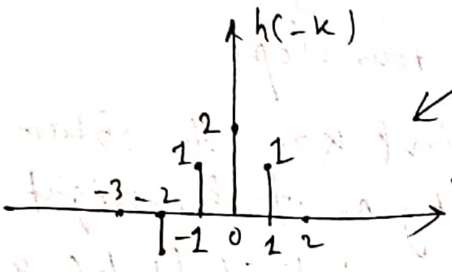




$n = n_0 = 0$

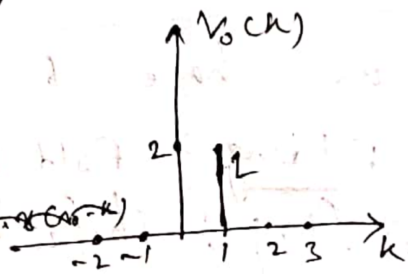
$V_0 = x(k) \cdot h(0-k)$

multiply



~~$V_0(k) = \sum_{k=}$~~

~~$x(k) \cdot h(0-k)$~~



$V_0(k) = \{0, 0, \underset{\uparrow}{2}, 2, 0, 0\}$

Sum of all terms in the product Sequence $= 4$.

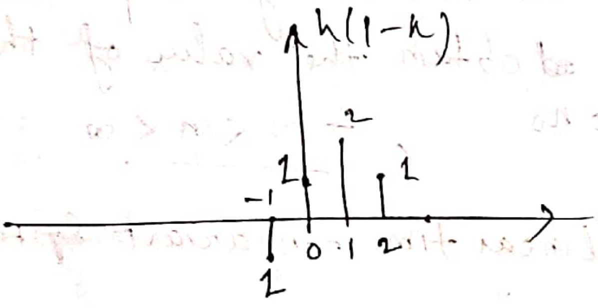
Then continue $n = 1$

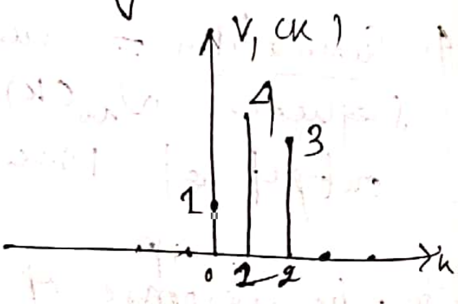
$$y(1) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(1-k)$$

~~shift by n_0 $h(-(k-1))$~~

~~$= \left(\sum_{k=-\infty} x(k)\right.$ ...~~

$n_0 = +1$ the $h(-k)$ shift by 1 to right



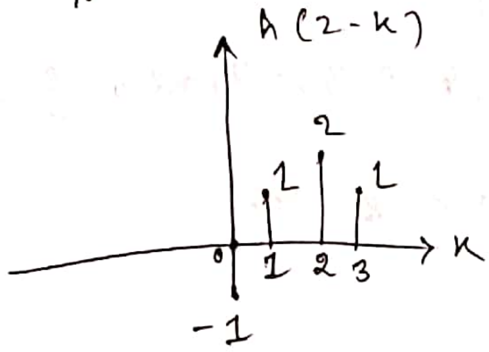


Then $V_1(k) = x(k) \cdot h(1-k)$

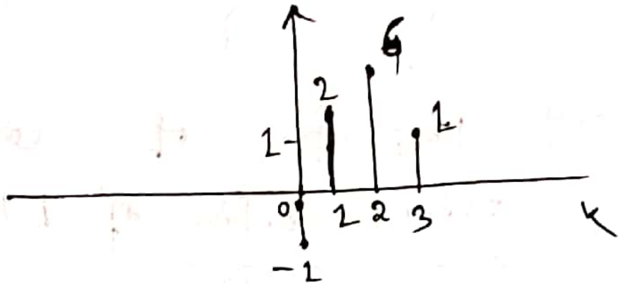
Sum ~~of~~ of all terms in the product Sequence

$= 8$

$n_0 = 2$.



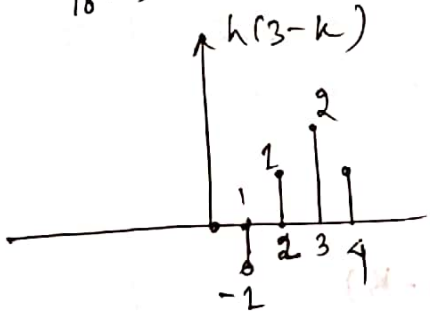
$V_2(n) = x(k) \cdot h(2-k)$



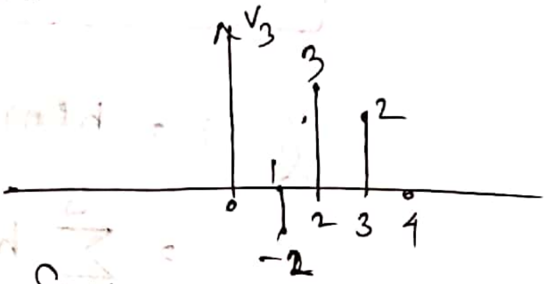
Sum of all terms in the product sequen

$= 9 - 1 = 8$.

$n_0 = 3$

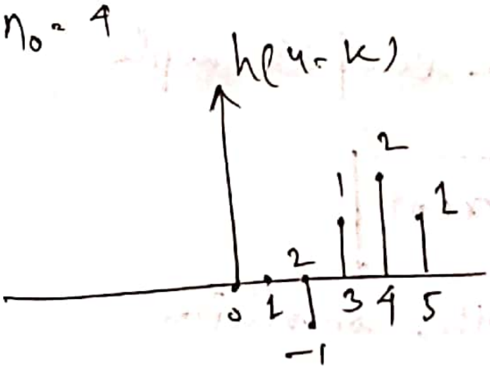


$V_3(k) = x(k) \cdot h(3-k)$

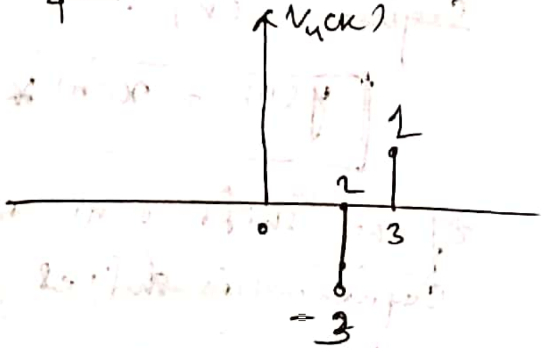


Sum = 3

$n_0 = 4$

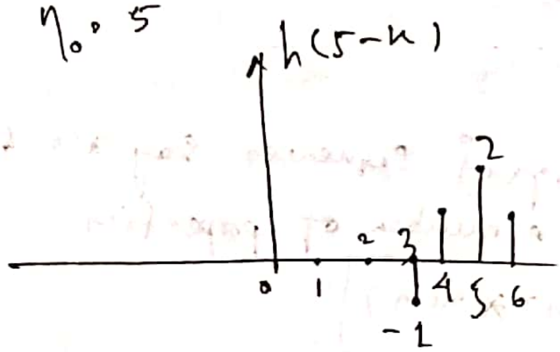


$V_4(k) = x(k) \cdot h(4-k)$

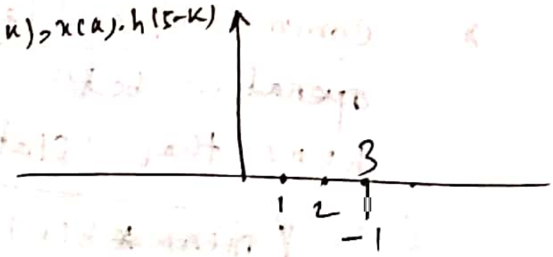


Sum = -2.

$n_0 = 5$

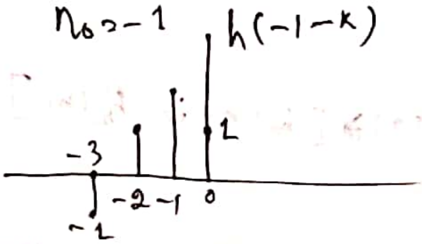


$V_5(k) = x(k) \cdot h(5-k)$

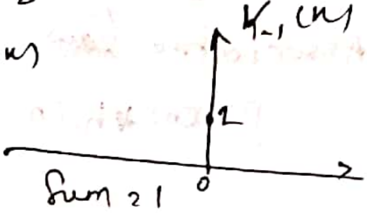


Sum = -1.

$n_0 = -1$



$V_{-1}(k) = x(k) \cdot h(-1-k)$



Sum = 1

Now we have the entire response.

$$y(n) = \{ \ldots 0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, \ldots \}$$

2.3.4 properties of convolution & the interconnection of LTI

$$\boxed{y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)}$$

$$y(n) = h(n) * x(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)$$

* identity & shifting properties

we also note that the unit sample sequence $\delta(n)$.

$$\boxed{y(n) = x(n) * \delta(n) = x(n)}$$

if we shift $\delta(n)$ by k, the convolution sequence is shifted also by k.

$$\boxed{x(n) * \delta(n-k) = y(n-k) = x(n-k)}$$

* commutative law

operation betn two signal sequences say $x(n)$ & $h(n)$ that satisfies a number of properties.

$$\boxed{x(n) * h(n) = h(n) * x(n)}$$

* Associative law

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

Distributive law

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

2.3.7 System with Finite Duration & infinite-Duration Impulse response

→ .a linear time-invariant System in term of it's impulse response h(n). it is also convenient however to subdivide the class of linear time-invariant System in to two types finite-duration impulse-response (FIR) & that have an infinite-duration impulse response (IIR).

For Causal FIR System

$h(n)$, $n < 0$ & $n \geq M$

$$\boxed{y(n) = \sum_{k=0}^{M-1} h(k) \cdot x(n-k)}$$

for ~~Causal~~ IIR

$$\boxed{y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)}$$

2.4 Discrete-Time System Described by Difference Equations

we have treated as linear & time-invariant System unit Sample response h(n) & to determine output y(n) of the ~~Sample of~~ system for any given input Sequence x(n),

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)}$$

FIR System is readily implemented by convolution sum. & IIR however it's practical implementation by convolution sum.

2.4.1 Recursive & Non recursive Discrete - Time system

Suppose that we wish to compute the cumulative average of a signal $x(n)$ in the interval $0 \leq k \leq n$.

$$y(n) = \frac{1}{n+1} \sum_{k=0}^{n} x(k) \quad , n = 0, 1, \ldots$$

the computation of $y(n)$ requires the storage of all the input samples $x(k)$ for $0 \leq k \leq n$.

→ $y(n)$ can be computed more efficiently by utilizing the previous o/p value $y(n-1)$

$$(n+1)\, y(n) = \sum_{k=0}^{n-1} x(k) + x(n)$$

$$= n\, y(n-1) + x(n)$$

$$\boxed{y(n) = \frac{n}{n+1} y(n-1) + \frac{1}{n+1} x(n)}$$

2.4.9 The impulse response of a linear - Time Invariant Recursive system

→ 1st order recursive system, the zero - state response given

$$y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k)$$

when $x(n) = \delta(n)$.

$$y_{zs}(n) = \sum_{k=0}^{n} a^k \delta(n-k)$$

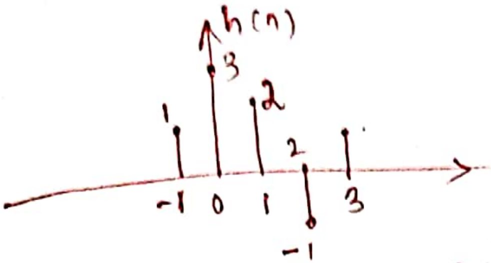
$$= a^n, \quad n \geq 0$$

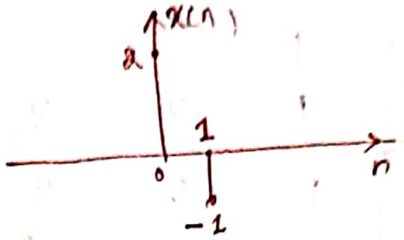
Hence $h(n) = a^n u(n)$

for Linear time invariant system

$$y_{zs}(n) = \sum_{k=0}^{n} h(k)\cdot x(n-k) \quad n \geq 0$$

Q $x(n) = 2\delta(n) - \delta(n-1)$, $h(n)$ is as shown in fig





Calculate convolution Sum.

Q.) Find the convolution of 2 sequence $x_1(n)$ & $x_2(n)$.

$x_1(n) = \{\underset{\uparrow}{1}, 2, 3\}$, $x_2(n) = \{\underset{\uparrow}{2}, 1, 4\}$

Sol^n Length of $x_1(n) = L_1 = 3$

$x_2(n) = L_2 = 3$

Length of $y(n) = L_1 + L_2 - 1 = 3+3-1 = 5$

$x_2(n)$ \ $x_1(n)$	1	2	3		
→ 2	2	4	6		
1	x	1	2	3	
4	x	x	4	8	12
	2	5	12	11	12

$y(n) = \{\underset{\uparrow}{2}, 5, 12, 11, 12\}$

Q $x_1(n) = \{\underset{\uparrow}{1}, 2, 3\}$

$x_2(n) = \{1, 2, \underset{\uparrow}{3}, 4\}$

Convolute the two sequence.

Solⁿ

$y(n) = x_1(n) * x_2(n)$

$x_2(n)$ \ $x_1(n)$	$\underset{}{\downarrow}$ 1	2	3			
1	1	2	3			
2	x	2	4	6		
→ 3	x	x	3	6	9	
4	x	x	x	4	8	12
	1	4	10	16	17	12

$y(n) = \{1, 4, \underset{\uparrow}{10}, 16, 17, 17\}$

Q obtain the graphical convolution of a discrete Linear time invariant System for input $x(n)$ shown in fig. The System impulse response is $h(n)$ as shown in fig.

$x(n) = \{-1, 1, 0, 1, -1\}$, $h(n) = \{1, \underset{\uparrow}{2}, 3\}$

CHAPTER -3

The Z - Transform & it's application to the analysis of LTI System.

3.1 The Z-Transform:-

→ The Z-transform play the Same role in the analysis of discrete - time Signal & LTI System. in the frequency domain.

→ In Z- domain, the convolution of 2-time domain Signal is equal to the multiplication of their corresponding Z-transform.

$$\boxed{X(Z) = \sum_{-\infty}^{\infty} x(n)\, z^{-n}} \quad — ①$$

where Z is a complex variable

$$\boxed{Z = re^{j\omega}}$$

$$X(Z) = Z\{x(n)\}$$

relationship betn x(n) & X(Z)

$$x(n) \overset{Z}{\longleftrightarrow} X(Z)$$

Region of Convergence:-

The region of convergence of X(Z) is all the Set of all values of Z for which X(Z) attains a finite value.

$-\infty$ to ∞ = Two Sided Z-Transform

Ex → $x(n) = \{-1, \underset{\uparrow}{2}, 1\ 3, 4\}$

0 + ∞ = one Sided Z - Transform.

Ex → $x(n) = \{\underset{\uparrow}{1}, 0, 3, -1, 2\}$

Ex

Q Find of Causal Sequence in to Z- Transform.

$x(n) = \{\underset{\uparrow}{1}, 0, 3, -1, 2\}$ → (Right hand Side Signal)

Ans

$$X(Z) = \sum_{-\infty}^{\infty} x(n) \cdot z^{-n} = \sum_{0}^{4} x(n) \cdot z^{-n}$$

$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$

$$= 1 + 0 \times z^{-1} + 3 \cdot z^{-2} + (-1)z^{-3} + 2z^{-4}$$

$$= 1 + 3z^{-2} - z^{-3} + 2z^{-4}$$

ROC: All values of z except zero, is the region of convergence.

Q)

$x(n) = \{-3, -2, -1, 0, \underset{\uparrow}{1}\}$ → left hand Sidel Signal

Find Z-Transform.

An

$$X(z) = \sum_{-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{-\infty}^{0} x(n) \cdot z^{-n}$$

$$= x(0) \cdot z^{0} + x(-1) z^{1} + x(-2) z^{2} + x(-3) z^{3} + x(-4) z^{4}$$

$$= 1 + 0 \times z - 1 \cdot z^{2} - 2z^{3} - 3z^{4}$$

$$= 1 + -z^{2} - 2z^{3} - 3z^{4}$$

ROC: All value of z except z = ∞, is the region of convergence.

Q) $x(n) = \{2, -1, 3, 2, \underset{\uparrow}{1}, 0, 2, 3, -1\}$ → (Two Sided Signal)

An

$$X(z) = \sum_{-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{-4}^{4} x(n)\, z^{-n}$$

$$= x(-4) \cdot z^{4} + x(-3) z^{3} + x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1} + x(2) \cdot z^{-2} + x(3) z^{-3} + x(4) \cdot z^{-4}$$

$$= 2 \cdot z^{4} + (-1) \cdot z^{3} + 3z^{2} + 2z^{1} + 1 \cdot z^{0} + 0 \cdot z^{-1} + 2z^{-2} + 3z^{-3} + (-1) z^{-4}$$

$$= 2z^{4} - z^{3} + 3z^{2} + 2z + 1 + 2z^{-2} + 3z^{-3} - z^{-4}$$

ROC: All values of z except z = 0 & z = ∞.

* Z-Transform of infinite duration causal sequence

Q)

$$x(n) = a^{n} u(n)$$

$$X(z) = \sum_{-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{0}^{\infty} a^n z^{-n}$$

$$= \sum_{0}^{\infty} (az^{-1})^n \qquad \text{Let} \; \therefore az^{-1} = x$$

$$= \frac{1}{1 - az^{-1}} , \text{ for } 0|x| < 1$$

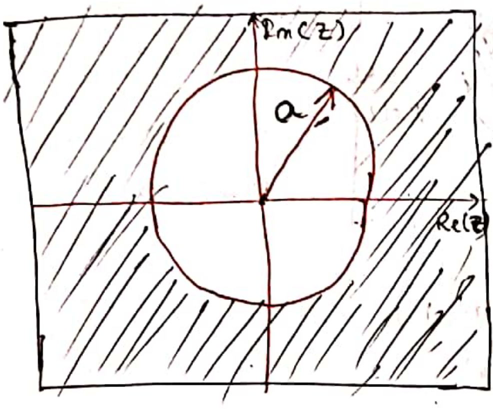
$$= \frac{z}{z-a}$$

$\therefore$ ROC : $|x| < 1$

$|az^{-1}| < 1$

$a < z$

$|z| > |a|$



⊛ $x(n) = -b^n u(-n-1)$ → non causal sequence or Anti causal sequence.

Ans

$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$$

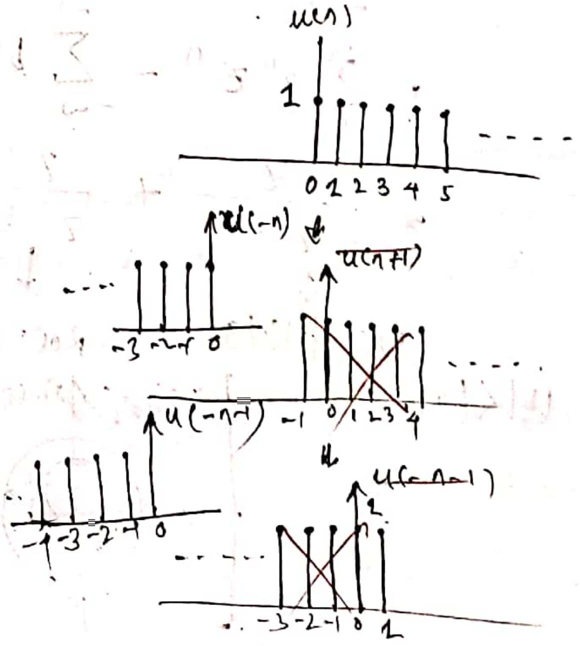
$$= \sum_{-\infty}^{\infty} -b^n u(-n-1) z^{-n}$$

$$= \sum_{-\infty}^{-1} -b^n z^{-n}$$

$$= -\sum_{-\infty}^{-1} (b z^{-1})^n$$

= ~~$\frac{b}{1-bz}$~~

$$= -\left[b^{-1} z^{+1} + b^{-2} z^{+2} + b^{+3} + b^{-\infty} z^{+\infty} \right]$$



$$= -\left[b^{-1} z^{1} + b^{-2} z^{2} + b^{-3} z^{3} \ldots\ldots + 0 \right]$$

$$= -\left[\sum_{0}^{-\infty} b^{n} z^{-n} - b^{0} z^{0} \right]$$

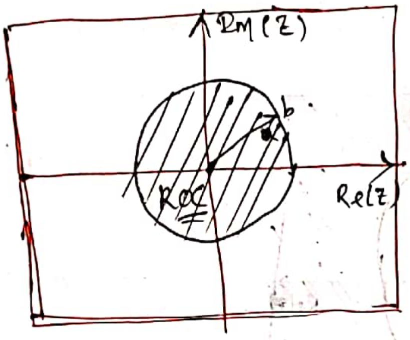
$$= -\left[\sum_{0}^{-\infty} (b^{-1} z)^{n} - 1 \right]$$

$$= -\left[\frac{1}{1 - b^{-1} z} - 1 \right]$$

$$= \frac{z}{z - b}$$

∴ ROC of this sequence is $|b^{-1} z| < 1$

$|z| < |b|$



Q $x(n) = a^{n} u(n) - b^{n} u(-n-1)$

Ans $X(Z) = \sum_{-\infty}^{\infty} x(n) \cdot z^{-n}$

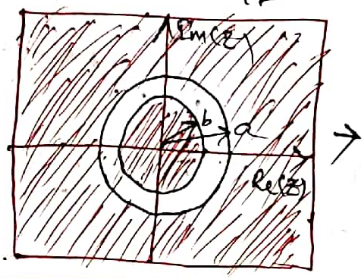
$$= \sum_{-\infty}^{\infty} \{ a^{n} u(n) - b^{n} u(-n-1) \} z^{-n}$$

$$= \sum_{0}^{\infty} a^{n} z^{-n} - \sum_{-\infty}^{-1} b^{n} z^{-n}$$

$$= \frac{z}{z - a} + \frac{z}{z - b}$$

↓ ROC: $|z| > |a|$ ↓ ROC: $|z| < |b|$

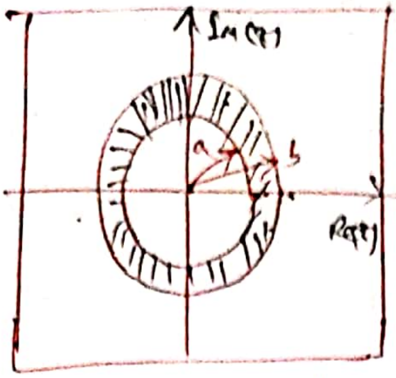
∴ if $|b| < |a|$



→ $|b| > |z| > |a|$

if $|a| < |b|$

ROC: $|a| < |z| < |b|$



3.1.2 The inverse Z-Transform

we have the Z-Transform $X(z)$ of a Signal & we must determine the Signal Sequence. The procedure for Transforming from the Z-domain to the time domain is called inverse Z-Transform.

we know

$$X(z) = \sum_{K=-\infty}^{\infty} x(n) \cdot z^{-n}$$

∴ Suppose z^{n-1} multiply both sides & integrate both sides

$$\oint_C X(z) z^{n-1} dz = \oint_C \sum_{K=-\infty}^{\infty} x(K) \cdot z^{n-1-K} dz$$

$$\boxed{x(n) = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz}$$

*** Stability & ROC**

→ let $h(n)$ is an impulse function of a causal or non-causal linear time invariant system ~~H~~ $H(z)$ be the System function then the stability of the system can be found from ROC in the the following theorem.

A System is said to be BIBO stable.

If all the poles of the System function lies inside the unit circle or ROC contains in the in unit circle.

$$h(n) = 2^n u(n), \quad \sum_{-\infty}^{\infty} h(n) < \infty$$

The series cannot convert, so it is unstable

$$H(z) = \sum_{-\infty}^{\infty} 2^n u(n) \cdot z^{-n}$$

$$= \sum_{0}^{\infty} 2^n \cdot z^{-n}$$

$$= \sum_{0}^{\infty} (2z^{-1})^n$$

$$= \frac{z}{z-2}$$ ∴ ROC: $|z| > 2$ doesn't contain in the unit circle.

Properties of ROC :-

→ ROC is a rim or disc in the z plane centered at origin.

→ ROC must be a connected region.

→ ROC doesn't contain any pole.

→ If $x(n)$ is a causal sequence, then ROC is all the values of z except $z=0$.

→ If $x(n)$ is a non-causal sequence, then ROC is all the values of z except $z = \infty$.

→ ROC of any linear time invariant contains unit circle..

3.2 Properties of Z-Transform :-

(1) Linearity :-

if $X(z) = Z[x(n)]$

$$Z[a_1x_1(n) + a_2x_2(n)] = a_1X_1(z) + a_2X_2(z).$$

→ $X_1(z) = Z[x_1(n)]$

$X_2(z) = Z[x_2(n)]$

Proof $Z[a_1x_1(n) + a_2x_2(n)]$

$$= \sum_{-\infty}^{\infty} [a_1x_1(n) + a_2x_2(n)] z^{-n}$$

$$= \sum_{-\infty}^{\infty} a_1 x_1(n)\cdot z^{-n} + \sum_{-\infty}^{\infty} a_2 x_2(n)\cdot z^{-n}$$

$$= a_1 X_1(z) + a_2 X_2(z)$$

(2) Time shift

if $X(z) = Z[x(n)]$

$$Z[x(n-m)] = z^{-m} X(z)$$

Proof

$Z[x(n-m)]$ $\quad\therefore l = n-m$, $n = l+m$

$$= \sum_{-\infty}^{\infty} x(n-m) z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l)\cdot z^{-(l+m)}$$

$$= \sum_{l=-\infty}^{\infty} z^{-m} x(l)\cdot z^{-l}$$

$$= z^{-m} \sum_{l=-\infty}^{\infty} x(l)\cdot z^{-l}$$

$$= z^{-m}\cdot X(Z)$$

(3) Time Reversal

if $X(z) = Z[x(n)]$

$$Z[x(-n)] = X(z^{-1})$$

Proof

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n} \quad \therefore l = -n$$

$$= \sum_{l=-\infty}^{\infty} x(l) z^{l}$$

$$= \sum_{l=-\infty}^{\infty} x(l)(z^{-1})^{-l}$$

$$= X(z^{-1})$$

Q Find the z-transform of δ(n).

$x(n) = \delta(n)$

Ans

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} z^{-0}$$

$$= 1$$ RoZ all Z

Q Find the Z-Transform of

$x(n) = u(n)$

Ans

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{n}$$

$$= \sum_{n=-\infty}^{\infty} u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$ let ∴ $a = z^{-1}$

$$= \sum_{n=0}^{\infty} a^{n}$$

$$= \frac{1}{1-a}$$ ∴ For $|a| < |1|$

$$= \frac{1}{1-z^{-1}}$$ ROC: ∴ $|z^{-1}| < |1|$

$|z| > 1.$

④ Scaling in Z-domain :-

$x(n) \xleftrightarrow{Z} X(Z)$ ROC : $r_1<|Z|<r_2$

then

$a^n x(n) \longleftrightarrow X(a^{-1}z)$ ROC: $|a|r_1<|z|<|a|r_2$

for any constant a, real or complex.

Proof

$$Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{n}$$

$$= X(a^{-1}z)$$

Question

1. Determine the Z-Transform & the ROC of the signal.

$$x(n) = [3(2^n) - 4(3^n)]u(n)$$

Ans If we define the signals.

$x_1(n) = 2^n u(n)$

$x_2(n) = 3^n u(n)$ — $x(n) = 3x_1(n) - 4x_2(n)$

According to linearity proper

$$X(Z) = 3X_1(Z) - 4X_2(Z)$$

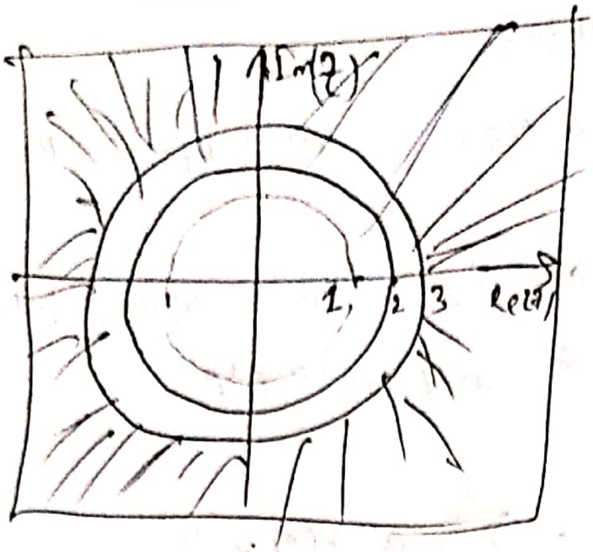
we know

$a^n u(n) \xleftrightarrow{Z} \frac{1}{1-az^{-1}}$ ROC: $|Z|>|a|$

$x_1(n) = 2^n u(n) \xleftrightarrow{Z} X_1(Z) = \frac{1}{1-2z^{-1}}$ ROC: $|Z|>2$

$x_2(n) = 3^n u(n) \xleftrightarrow{Z} X_2(Z) = \frac{1}{1-3z^{-1}}$ ROC: $|Z|>3$

$$X(Z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$$



ROC: $|z| > 2$ is because ROC doesn't include any pole.

So

ROC: $|z|>3$

Q. Determine the Z-Transform of the signals

(a) $x(n) = (\cos w_0 n) u(n)\}$

(b) $x(n) = (\sin w_0 n) u(n)\}$

Ans

(a) $x(n) = \cos w_0 n \, u(n)$

we know euler formul

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Sol

$x(n) = \cos w_0 n \, u(n)$

$$= \frac{1}{2} e^{jw_0 n} . u(n) - \frac{1}{2} e^{-jw_0 n} . u(n)$$

$$X(z) = \frac{1}{2} Z\{e^{jw_0 n} . u(n)\} + \frac{1}{2} Z\{e^{-jw_0 n} . u(n)\}$$

So we know

$$\frac{1}{2} e^{jw_0 n} . u(n) \xleftrightarrow{Z} \frac{1}{1 - e^{jw_0} z^{-1}} \quad ROC: |z|>1$$

$$e^{-jw_0 n} u(n) \xleftrightarrow{Z} \frac{1}{1 - e^{-jw_0} z^{-1}} \quad ROC: |z|>1$$

$$X(z) = \frac{1}{2} \frac{1}{1 - e^{jw_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-jw_0} z^{-1}}$$

$$= \frac{1}{2} \left[\frac{1 - e^{-jw_0} z^{-1} + 1 - e^{jw_0} z^{-1}}{(1 - e^{jw_0} z^{-1})(1 - e^{-jw_0} z^{-1})} \right)$$

$$= \frac{1}{2}\left[\frac{2 - e^{-j\omega_0}z^{-1} + e^{j\omega_0}z^{-1}}{1 - e^{-j\omega_0}z^{-1} - e^{j\omega_0}z^{-1} + e^{j\omega_0}z^{-1}\cdot e^{-j\omega_0}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - 2\left[\frac{e^{j\omega_0}\cancel{z^{0}} + e^{-j\omega_0}\cancel{z^{0}}}{2}\right]z^{-1}}{1 - e^{-j\omega_0}z^{-1} - e^{j\omega_0}\cdot z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - 2(\cos\omega_0)z^{-1}}{1 - (-2)\left[\frac{e^{j\omega_0} + e^{-j\omega_0}}{2}\right]z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}(1 - z^{-1}\cos\omega_0)}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}\right]$$

$$= \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}} \qquad \text{ROC} : |z| > 1$$

b) $x(n) = \sin\omega_0 n \cdot u(n)$

$\therefore \sin\theta = \frac{e^{j\theta} - e^{j\theta}}{2j}$

④ Multiplication property or Scaling

If $x(n) \xleftrightarrow{Z} X(Z)$, ROC : $r_1 < |Z| < r_2$

$a^n x(n) \xleftrightarrow{Z} X(a^{-1}z)$ ROC: $|a| r_1 < |Z| < |r_2| a$

Proof

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \, (a^{-1}z)^{-n}$$

$$= X \cdot (a^{-1}z).$$

Q a) $x(n) = a^n (\cos w_0 n) u(n)$

b) $x(n) = a^n (\sin w_0 n) u(n)$.

Determine the Z - Transform.

Ans

b) $x(n) = a^n (\sin w_0 n) \cdot u(n)$

Simplify

$x(n) = \sin w_0 n \cdot u(n)$

$$X(Z) = \frac{Z^{-1} \sin w_0}{1 - 2a Z^{-1} \cos w_0 + Z^{-2}}$$

apply multiply propert

$$= \frac{a Z^{-1} \sin w_0}{1 - 2aZ^{-1} \cos w_0 + a^2 Z^{-2}} \quad |Z| > |a|$$

Q Determine the Z- Transform of the Signal

$x(n) = u(-n)$

Ans $u(n) \xleftrightarrow{Z} \frac{1}{1 - Z^{-1}}$ ROC: $|Z| > 1$

$u(-n) \xleftrightarrow{Z} \frac{1}{1 - (Z^{-1})^{-1}}$ ROC $|Z| < 1$

$$= \frac{1}{1 - Z}$$

⑤ Differentiation in the z-domain

$$x(n) \xleftrightarrow{z} X(z)$$

$$n\,x(n) \xleftrightarrow{z} -z\frac{dX(z)}{dz}$$

proof

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n)\cdot(-n)\, z^{-n-1}$$

$$= -z^{-1}\sum_{n=-\infty}^{\infty}[n\,x(n)]z^{-n}$$

$$= -z^{-1}\{n\,x(n)\}$$

Q Determine the Z-Transform of the Signal

$$x(n) = n a^n u(n)$$

Ans

we know

$$x_1(n) = a^n u(n) \Rightarrow X_1(z) = \frac{1}{1-az^{-1}} \quad \text{ROC: } |z|>|a|$$

then

$$n a^n u(n) \xleftrightarrow{z} X(z) = -z\frac{dX_1(z)}{dz} = \qquad \frac{d}{dt}\frac{x}{y} = \frac{x'y - xy'}{y^2}$$

$$= \cancel{\frac{0-(-a)\cdot z^{-2}}{(1-az^{-1})^2} = \frac{az^{2}}{}}$$

$$= \frac{0-a(-1)z^{-2}}{(1-az^{-1})^2} = \frac{az^{-1}}{(1-az^{-1})^2} \qquad \text{ROC: } |z| > a$$

⑥ convolution of two Sequence

of

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

$$x(n) = x_1(n) * x_2(n) \xleftrightarrow{z} X(z) = X_1(z)\cdot X_2(z)$$

Ex compute the convolution x(n) of the Signal.

$$x_1(n) = \{1, -2, 1\}$$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{else where} \end{cases}$$

Ans

$$x_1(n) = \{1, -2, 1\} \quad x_2(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1, 1\}$$

$$x(n) = x_1(n) * x_2(n)$$

$X(z) = X_1(z) \cdot X_2(z)$

$X_1(z) = 1 - 2z^{-1} + z^{-2}$

$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$
$$- 2z^{-1} - 2z^{-2} - 2z^{-3} - 2z^{-4} - 2z^{-5} - 2z^{-6}$$
$$+ z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7}$$

$$= 1 - z^{-1} - 2z^{-6} + z^{-7}$$

$x(n) = \{ \underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1 \}$

⑥ Correlation of two Sequence

if $x_1(n) \xleftrightarrow{Z} X_2(z)$

$x_2(n) \xleftrightarrow{Z} X_2(z)$

then $r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2(n-l) \xleftrightarrow{Z}$

$R_{x_1x_2}(z) = X_1(z) \cdot X_2(z^{-1})$

or $\boxed{r_{x_1x_2}(l) = x_1(l) * x_2(-l)}$ Auto Correlation

Cross - Correlation

$\boxed{r_{xy}(l) = x(l) * y(-l)}$ ✓

Q Determine the autocorrelation sequence of the Signal.

$x(n) = a^n u(n) \quad -1 < a < 1$

Ans

Auto correlation convolution

$R_{xx}(z) = Z\{r_{xx}(l)\} = X(z) \cdot X(z^{-1})$

$= x_1(n) * x_2(-n))$

$X(z) = \frac{1}{1 - az^{-1}}$ ROC: $|z| > |a|$

$x(-n) \xleftrightarrow{Z} X(z^{-1})$

$= \frac{1}{1 - az}$ ROC: $|z| < \frac{1}{|a|}$

anticau

$$R_{xx}(z) = \frac{1}{1-az^{-1}} \cdot \frac{1}{1-az}$$

$$= \frac{1}{1-a(z+z^{-1})+a^2} \quad , \text{ROC: } |a| < |z| < \frac{1}{a}$$

⑦ multiplication of two sequence

If $x_1(n) \xleftrightarrow{z} X_1(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$

$$x(n) = x_1(n) \cdot x_2(n) \xleftrightarrow{z} X(z) = \frac{1}{2\pi j}\oint_c X_1(v) \cdot X_2\left(\frac{z}{v}\right) v^{-1} dv$$

⑧ Parseval's Relation

if $x_1(n)$ & $x_2(n)$ are complex valued sequence

$$\boxed{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j}\oint X_1(v) \cdot X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv}$$

⑨ The Initial value theorem

If $x(n)$ is causal [ie $x(n)=0$ for $n<0$], then

$$\boxed{x(0) = \lim_{z\to\infty} X(z)}$$

⑩ The final value theorem

$$\boxed{x(\infty) = \lim_{z\to 1} (z-1)\, X_T(z)}$$

Some common Z-Transform

	Signal $x(n)$	Z-Transform $X(z)$	ROC
1.	$\delta(n)$	1	All z
2.	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3.	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4.	$n a^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5.	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6.	$-n a^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$

7.	$\cos \omega_0 n \cdot u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert>1$
8.	$\sin\omega_0 n \cdot u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert>1$
9.	$u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert<1$

3.3 Rational Z-Transform

family of Z-Transform are those for which $X(Z)$ is a rational, that is a ratio of two polynomials in z^{-1} (or z).

3.3.1 poles & zeros

The zeros of a Z-Transform $X(Z)$ are the values of z for which $X(Z)=0$. The poles of a Z-Transform are the values of z for which $X(Z)=\infty$ if the $X(Z)$ is a rational function.

$$X(Z) = \frac{B(Z)}{A(Z)} = \frac{b_0+b_1z^{-1}+\cdots\cdots+b_mz^{-M}}{a_0+a_1z^{-1}+\cdots\cdots+a_Nz^{-N}}$$

$$= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

if $a_0\neq 0$ $b_0\neq 0$ we can avoid the negative power of z by factoring out $b_0 z^{-m}$ & $a_0 z^{-N}$

$$= \frac{b_0 z^{-M} z^{M} + (b_1/b_0)z^{M-1} + \cdots + b_M/b_0}{a_0 z^{-N} z^{N} + (a_1/a_0)z^{N-1} + \cdots\cdots + a_N/a_0}$$

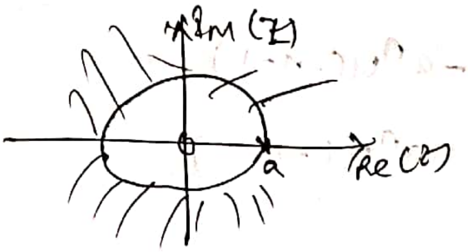
$$X(Z) = \frac{b_0}{a_0} z^{-M+N} \frac{(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

we represent graphically pole (X) & Zeros (0).

Eg

$x(n) = a^n u(n)$

Sol



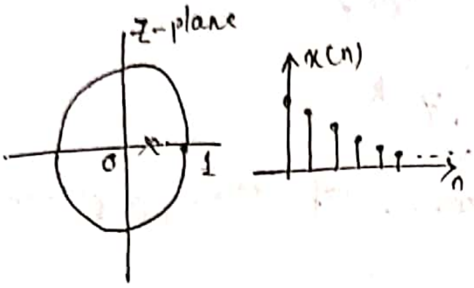
$$X(Z) = \frac{Z}{Z-a}$$

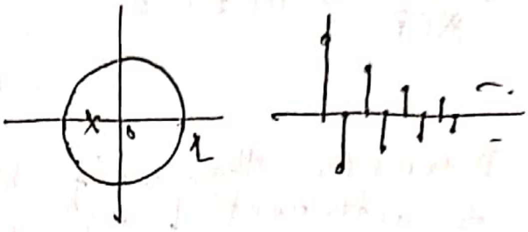
zero = $z=0$

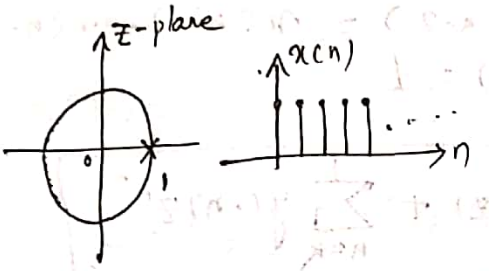
pole = $z=a$ Roc : $\lvert z\rvert>\lvert a\rvert$

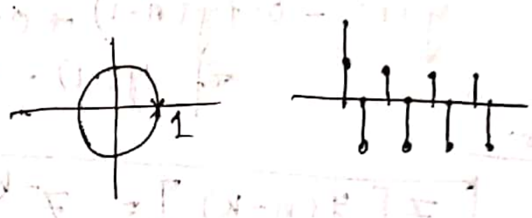
3.3.2 pole location & Time-Domain Behaviour for Causal Signals

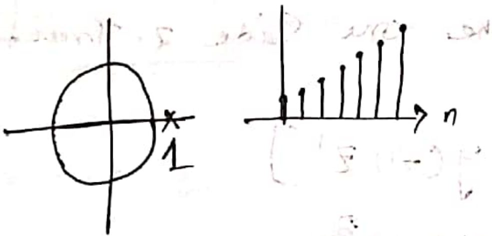
we deal exclusively with real, causal signal. we see that the characteristic behavior of causal signal depends on whether the poles of the transform are required $|z|<1$ in the region $r>1$ or on unit the circle. the $z=1$ radius of 1 it is called unit circle.

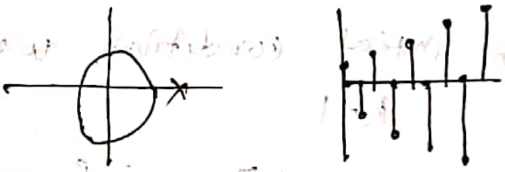












3.3.3 The System function of a linear Time-invariant System

The input sequence $x(n)$ can be obtained by computing the convolution of $x(n)$ with the unit sample response of the system. it expressed in z-domain

$$Y(z) = H(z) \cdot X(z)$$

$$x(n) \xleftrightarrow{z} X(z)$$
$$y(n) \xleftrightarrow{z} Y(z)$$
$$h(n) \xleftrightarrow{z} H(z)$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)}}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

A system function is described by linear time constant co-efficient differential equation.

$$y(n) = -a_k \sum_{k=1}^{N} y(n-k) + b_k \sum_{k=0}^{M} x(n-k)$$

$$Y(z) = -a_k \sum_{k=1}^{N} z^{-k} Y(z) + b_k \sum_{k=0}^{M} z^{-k} X(z)$$

$$Y(z)\left[1 + a_k \sum_{k=1}^{N} z^{-k}\right] = b_k \cancel{X(z)} \sum_{k=0}^{M} z^{-k} X(z).$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{b_k \sum_{k=0}^{M} z^{-k} X(z)}{1 + a_k \sum_{k=1}^{N} z^{-k}}$$

Q. Determine the unit step response of the system to differential eqn is

$$y(n) - 0.7y(n-1) + 0.12y(n-2) = x(n-1) + x(n-2)$$

if $y(-1) = y(-2) = 1$

Ans.

$$\boxed{Z[y(n-k)] = z^{-k}\left[Y(z) + \sum_{n=k}^{1} y(-n) z^{n}\right]}$$

for initial condition we take one side z-Transform

k=1

$$Z[y(n-1)] = z^{-1}[Y(z) + y(-1) z^{1}]$$

$$= z^{-1}\{Y(z) + y(-1)z\}$$

$$Z[y(n-2)] = z^{-2}\left[Y(z) + \sum_{2}^{1} y(-n) z^{n}\right]$$

$$= z^{-2}[Y(z) + y(-2)z^{2} + y(-1)z]$$

$$= z^{-2}Y(z) + y(-2) + y(-1)z^{-1}$$

Put value

$$\rightarrow Y(z) - 0.7[z^{-1}Y(z) + y(-1)] + 0.12[z^{-2}[Y(z) + y(-2)z^{2} + y(-1)z^{1}]]$$

$$= \cancel{Y(z)[1 - 0.7z^{-1} + 0.7 + 0.12z^{-2} + 1 + z^{-1}]}$$

$$= Y(z) - 0.7z^{-1}Y(z) - 0.7 + 0.12z^{-2}Y(z) + 0.12y(-2) + 0.12y(-1)z^{-1}$$

$$= Y(z)[1 - 0.7z^{-1} + 0.12z^{-2}] - 0.7 + 0.12 + 0.12z^{-1}$$

Q Determine the system function & the unit sample response of the system described by the differential equation.

$$y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

An By computing the Z-Transform of the difference eq^n.

$$Y(Z) = \frac{1}{2} Z^{-1} Y(Z) + 2X(Z)$$

$$Y(Z) \left(1 - \frac{1}{2} Z^{-1}\right) = 2X(Z)$$

$$\frac{Y(Z)}{X(Z)} = \frac{2}{1 - 1/2 Z^{-1}}$$

$$H(Z) = \frac{2}{1 - 1/2 Z^{-1}}$$

$$h(n) = 2(1/2)^n u(n).$$

Q.4 Discuss Inverse Z Transform

$$x(n) = \frac{1}{2\pi j} \oint X(Z) . Z^{n-1} dZ$$

where the integral is a contour integral over a closed path c that encloses the origin & lies with the region of convergence of X(Z)

1. Direct evaluation of (3.4.1) by contour integration.
2. Expansion in to a series of term in the variables Z & Z^{-1}
3. partial fraction expansion & table look up.

Partial Fraction Expansion

in the table look up method.

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z) + \dots + a_K X_K(Z).$$

Inverse of X(Z) is x(n)

$$x(n) = a_1 x_1(n) + a_2 x_2(n) + \dots + a_K x_K(n)$$

with out loss of generality. $a_0 = 1$

$$X(Z) = \frac{B(Z)}{A(Z)}$$

$$= \frac{b_0 + b_1 z^{-1} + \cdots\cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots\cdots + a_N z^{-N}}$$

$a_N \neq 0$ & $M < N$

To Simplyfy our discussion we eliminate the negative power

$$X(Z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots\cdots + b_M z^{M-N}}{z^N + a_1 z^{N-1} + \cdots\cdots + a_N}$$

Which contain only positive power of z Since $N \geq M$

$$\boxed{\frac{X(Z)}{Z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots\cdots + b_M z^{M-N-1}}{z^N + a_1 z^{N-1} + \cdots\cdots + a_N}}$$

* Direct poles

poles of $P_1, P_2, \cdots P_N$

$$\boxed{\frac{X(Z)}{Z} = \frac{A_1}{Z - P_1} + \frac{A_2}{Z - P_2} + \cdots \frac{A_N}{Z - P_N}}$$

Q Determine the partial-fraction expansion of the proper function.

$$X(Z) = \frac{1}{1 - 1.5 Z^{-1} + 0.5 Z^{-2}}$$

A we eliminate negative power & we multiply by z^2 in numerator & denominator.

$$X(Z) = \frac{Z^2}{Z^2 - 1.5 Z + 0.5}$$

$$\frac{X(Z)}{Z} = \frac{Z}{(Z-1)(Z-0.5)} = \frac{A_1}{Z-1} + \frac{A_2}{Z-0.5}$$

$$z = A_1(z-0.5) + A_2(z-1)$$

$$A_1 = \cancel{(z-0.5)} \times \frac{z}{(z-1)\cancel{(z-0.5)}}\Bigg|_{z=0.5}$$

$$= \frac{0.5}{0.5-1} = \underline{\underline{-1}}$$

$$A_2 = (z-1) \times \frac{z}{(z-1)(z-0.5)}\Bigg|_{z=1}$$

$$= \frac{1}{1-0.5} = 2.$$

$$\frac{X(z)}{z} = \frac{-1}{z-1} + \frac{2}{z-0.5}$$

$$= \frac{-z}{z-1} + \frac{2z}{z-0.5}$$

$$\boxed{x(n) = -(1)^n u(n) + 2(1/2)^n u(n)}$$

multiorder poles

Ex Determine the partial fraction of

$$X(Z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

Sol

$$X(Z) = \frac{\cancel{z} \cdot z^2}{(z+1)(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

$$= \frac{A_1}{(z+1)} + \frac{A_2}{(z-1)} + \frac{A_3}{(z-1)^2}$$

$$\frac{z^2}{(z+1)(z-1)^2} = \frac{A_1(z-1)^2 + A_2(z+1)(z-1) + A_3(z+1)}{}$$

$$A_1 = (z+1)\frac{z^2}{(z+1)(z-1)^2}\Bigg|_{z=-1}$$

$$= \frac{1}{4}$$

$$\therefore\ A_2 = \frac{1}{(2-1)!}\ \frac{d}{dz}\left[(z-p_1)^2\frac{X(z)}{z}\right]_{z=p_1}$$

$$A_2 = \frac{1}{(2-1)!}\ \frac{d}{dz}\left[\cancel{\frac{z^2}{2}}\ (z-1)^2\ \frac{z^2}{(z+1)(z-1)^2}\right|_{z=1}$$

$$= \frac{d}{dz}\left[\frac{z^2}{z+1}\right|_{z=1}$$

$$= \frac{1}{1}\ \frac{2z(z+1)-(z^2 \cdot 1)}{(z+1)^2}\Bigg|_{z=1}$$

$$= \frac{2z^2+2z-z^2}{(z+1)^2}\Bigg|_{z=1}$$

$$= \frac{2+2-1}{4} = \boxed{\frac{3}{4}}$$

$$A_3 = (z-1)^2\ \frac{z^2}{(z+1)(z-1)^2}\Bigg|_{z=1}$$

$$= \frac{1}{2}$$

$$\frac{X(z)}{z} = \frac{1/4}{z+1} + \frac{3/4}{(z-1)} + \frac{1/2}{(z-1)^2}$$

$$= \frac{1/4\, z}{z+1} + 3/4\ \frac{z}{z-1} + 1/2\ z/(z-1)^2$$

$$= \frac{1}{4}(-1)^n u(n) + \frac{3}{4}(1)^n u(n) + 1/2\, n(1)^n u(n)$$

chapter-1 Discuss Fourier Transform & it's applications

~~Discrete Discrete~~

5.1~~5.1 frequency Domain Sampling~~ → The DFT

To perform frequency analysis on a dts freq we convert the time-domain sequence to an equivalent frequency-domain represenfation. is given by $X(\omega)$.

$\boxed{x(n) \xleftrightarrow{DTFT} X(e^{j\omega})}$ $\boxed{x(n) \xleftrightarrow{DFT} X(\omega)}$, $\boxed{x(t) \xleftrightarrow{CTFT} X(j\omega)}$

5.1 frequency - Domain Sampling & Reconstruction of DTS

---

an aperiodic discrete-time Signal $x(n)$ with Fourier transform.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} \longrightarrow \text{DTFT}$$

Suppose that we sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ radians betn successive samples. $X(\omega)$ is periodic with period 2π. we take N equidistant samples in the interval $0 \leq \omega \leq 2\pi$ with spacing $\delta\omega = 2\pi/N$.

$$\omega = \frac{2\pi k}{N}$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots N-1$$

$$= \cdots \sum_{n=-N}^{-1} x(n) \cdot e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$+ \sum_{n=N}^{2N-1} x(n) \cdot e^{-j2\pi kn/N} + \cdots$$

it can be expande in a Fourier Se.

$$x_p = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N} \quad n = 0, 1, \ldots N-1$$

fourier Series co-efficient

$$\boxed{C_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p e^{-j2\pi kn/N}} \quad k = 0, 1, \ldots N-1$$

Ex

Consider the Signal

$x(n) = a^n u(n)$

Determine DFT

Ans

$$X(w) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) . e^{-jwn}$$

$$= \sum_{n=0}^{\infty} a^n . e^{-jwn}$$

$$= \sum_{n=0}^{\infty} (a . e^{-jw})^n$$

$$= \frac{1}{1 - a e^{-jw}}$$

The DFT

The frequency Samples $X(2\pi K/N)$, $K = 0, 1, \dots N-1$ uniquely represent the finite-duration sequence $x(n)$. The 'L' equidistance Samples $X(w)$ are sufficed to reconstruct ~~x~~ form. padding the seq $x(n)$ with N-L zero &

Computing ~~N~~

N- point DFT

~~$X(K) = \sum$~~

finite duration: $X(w) = \sum_{n=0}^{L-1} x(n) . e^{-jwn}$ $0 \leq w < 2\pi$

$$X(K) = \sum_{n=0}^{N-1} x(n) . e^{-j2\pi kn/N} \quad , K = 0, 1, 2 \dots N-1$$

inverse = $\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}$

DFT

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N}} \quad , k = 0, 1, 2, 3 \cdot \cdot \cdot N-1.$$

IDFT :

$$\boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}} \quad , \quad n = 0, 1, 2, --- N-1$$

5.1.3 The DFT as a Linear Transform

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad k = 0, 1, ----- N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \quad n = 0, 1, ---- N-1$$

$$\boxed{W_N = e^{-j2\pi/N}} \qquad \boxed{W_N^k = e^{-j2\pi k/N}}$$

it is instructive to view the DFT & IDFT as linear Transformation on Sequence $\{x(n)\}$ & $\{x(k)\}$. let us define an N- point vector X_N of the signal sequence $x(n)$, $n = 0, 1, ----- N-1$ an N-point vector X_N of frequency Samples & an $N \times N$ matrix W_N as

$$X_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} , \quad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

with definition the N- point DFT may be expressed in matrix form

$$\boxed{X_N = W_N X_N} \qquad \text{DFT}$$

IDFT

$$x_N = \frac{1}{N} W_N^{-1} X_N$$

$$\boxed{x_N = \frac{1}{N} W_N^{*} X_N} \quad \text{—— ②}$$

where W_N^* denotes the complex conjugate of the matrix W_N.

The DFT formula

$$X_N = W_N x_N$$

$$x_N = W_N^{-1} X_N \quad \text{—— ③}$$

eq ② & ③ compare.

$$\boxed{W_N^{-1} = \frac{1}{N} W_N^{*}}$$

which, in turn, implies that

$$\boxed{W_N W_N^{*} = N I_N}$$

where I_N is an $N \times N$ identity matrix. Therefore the matrix W_N in the transformation is an Orthogonal (unitary) matrix.

Ex Compute the DFT of the four-point sequence.

$$x(n) = (0, 1, 2, 3)$$

Sol The first step is to determin the matrix W_4.

$$W_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

$$W_N^k = e^{\frac{-j2\pi k}{N}}$$

$$\rightarrow W_4^0 = e^0 = 1$$

$$W_4^1 = e^{\frac{-j2\pi \cdot 1}{4}} = e^{-j\pi/2} = \cos \pi/2 - i \sin \pi/2 = -j$$

$$W_4^2 = e^{\frac{-j2\pi \cdot 2}{4}} = e^{-j\pi} = \cos\pi - i\sin\pi = -1 - 0 = \cancel{0} -1$$

$$W_4^3 = e^{\frac{-j2\pi \cdot 3}{4}} = e^{-j3\pi/2} = \cos 3\pi/2 - j\sin 3\pi/2 = j$$

$$W_4^4 = e^{\frac{-j2\pi \cdot 4}{4}} = e^{-j2\pi} = \cos 2\pi - j\sin 2\pi = 1$$

$$W_4^6 = e^{\frac{-j2\pi \cdot 6}{4}} = e^{-j3\pi} = \cos 3\pi - j\sin 3\pi = -1$$

$$W_4^9 = e^{\frac{-j2\pi \cdot 9}{4}} = e^{-j9\pi/2} = \cos 9\pi/2 - j\sin 9\pi/2 = -j$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X_4 = W_4 \cdot x_4$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 0 + 1\cdot 1 + 1\cdot 2 + 1\cdot 3 \\ 1\cdot 0 + -j\cdot 1 + (-1)\cdot 2 + j\cdot 3 \\ 1\cdot 0 + (-1)\cdot 1 + 1\cdot 2 + (-1)\cdot 3 \\ 1\cdot 0 + j\cdot 1 + (-1)\cdot 2 + (-j)\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -j - 2 + j3 \\ -1 + 2 - 3 \\ j - 2 - 3j \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -2 + j2 \\ -2 \\ -2 - 2j \end{bmatrix}$$

Q Determine the 8-point DFT of the Sequence.

$x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$

Ans

$$W_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ 1 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ 1 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ 1 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ 1 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{25} \\ 1 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ 1 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{21} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix}$$

We know

$$W_N^K = e^{-j\frac{2\pi K}{N}}$$

$$W_8^1 = e^{-j\frac{2\pi \cdot 1}{8}}$$

$$= e^{-j\pi/4} = \cos\pi/4 - i\sin\pi/4$$

$$= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j\frac{2\pi \cdot 2}{8}}$$

$$= e^{-j\pi/2} = \cos\pi/2 - i\sin\pi/2 = \cancel{\tfrac{1}{\sqrt{2}}}\ \cancel{0}\ j\tfrac{1}{\sqrt{2}}$$

$$= \cancel{\tfrac{1}{\sqrt{2}}}\,0 - j\cancel{\tfrac{1}{\sqrt{2}}} = -j$$

$$W_8^3 = e^{-j\frac{2\pi \cdot 3}{8}} = e^{-j3\pi/4} = \cos 3\pi/4 - j\sin 3\pi/4$$

$$= -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^4 = e^{-j\frac{2\pi \cdot 4}{8}} = e^{-j\pi} = \cos\pi - i\sin\pi$$

$$= -1$$

$$W_8^5 = e^{-j\frac{2\pi \cdot 5}{8}} = e^{-j5\pi/4} = \cos 5\pi/4 - j\sin 5\pi/4 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$W_8^6 = e^{-j\frac{2\pi \cdot 6}{8}} = e^{-j3\pi/2} = \cos 3\pi/2 - j\sin 3\pi/2 = 0 - j(-1) = j$$

$$W_8^7 = e^{-j\frac{2\pi \cdot 7}{8}} = e^{-j7\pi/4} = \cos 7\pi/4 - j\sin 7\pi/4 = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$$

$$W_8^8 = e^{-j\frac{2\pi \cdot 8}{8}} = \cos 2\pi - j\sin 2\pi = \cancel{0}\ 1$$

$$W_8^9 = e^{-j\frac{8\pi \cdot 9}{84}} = e^{-j9\pi/4} = \cos 9\pi/4 - j\sin 9\pi/4 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^{10} = e^{-j\frac{2\pi \cdot 10}{8}} = e^{-j5\pi/2} = \cos 5\pi/2 - j\sin 5\pi/2 = -j$$

$$W_8^{12} = e^{-j\frac{2\pi \cdot 12}{8}} = e^{-j3\pi} = \cos 3\pi - j\sin 3\pi = -1$$

$W_8^{14} = e^{-j\frac{2\pi\cdot 14}{8}} = e^{-j7\pi/2} = \cos 7\pi/2 - j\sin 7\pi/2 = +j$

$W_8^{15} = e^{-j\frac{2\pi\cdot 15}{8}} = e^{-j15\pi/4} = \cos 15\pi/4 - j\sin 15\pi/4$

$= \frac{1}{\sqrt{2}} + j/\sqrt{2}$

$W_8^{18} = e^{-j\frac{2\pi\cdot 18}{8}} = -j$

$W_8^{21} = e^{-j\frac{2\pi\cdot 21}{8}} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$W_8^{20} = e^{-j\frac{2\pi\cdot 20}{8}} = e^{-j5\pi} = \cos 5\pi - j\sin 5\pi = -1$

$W_8^{24} = e^{-j\frac{2\pi\cdot 24}{8}} = e^{-6\pi} = 1$

$W_8^{25} = e^{-j\frac{2\pi\cdot 25}{8}} = e^{-j25\pi/4} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^{28} = e^{-j\frac{2\pi\cdot 28}{8}} = e^{-j7\pi} = -1$

$W_8^{30} = e^{-j\frac{2\pi\cdot 30}{8}} = e^{-j15\pi/2} = j$

$W_8^{35} = e^{-j\frac{2\pi\cdot 35}{84}} = e^{-j35\pi/4} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^{36} = -1$

$W_8^{42} = -j$

$W_8^{49} = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

Short cut

$$\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & \frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -j & & -1 & & -j & \\
3 & 1 & -j & -1 & j & 1 & -j & -1 & j \\
4 & 1 & & j & \frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -1 & & -j & \\
5 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
6 & 1 & & -j & & -1 & & j & \\
7 & 1 & j & -1 & -j & 1 & j & -1 & -j \\
8 & 1 & & j & & -1 & & -j &
\end{array}$$

Short Cut

$$X_8 = W_8 \cdot x_8$$

$$\begin{array}{c|cccccccc}
 & 1\checkmark & 2 & 3\checkmark & 4 & 5\checkmark & 6 & 7\checkmark & 8 \\
\checkmark 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\checkmark 2 & 1 & \frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}} & -j & -\frac{1}{2}-j/\sqrt{2} & -1 & -\frac{1}{\sqrt{2}}+j/\sqrt{2} & j & \frac{1}{\sqrt{2}}+j/\sqrt{2} \\
\checkmark 3 & 1 & -j & -1 & j & 1 & -j & -1 & j \\
4 & 1 & -\frac{1}{\sqrt{2}}-j/\sqrt{2} & j & \frac{1}{\sqrt{2}}-j/\sqrt{2} & -1 & \frac{1}{\sqrt{2}}+j/\sqrt{2} & -j & -\frac{1}{\sqrt{2}}+j/\sqrt{2} \\
\checkmark 5 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
6 & 1 & -\frac{1}{\sqrt{2}}+j/\sqrt{2} & -j & \frac{1}{\sqrt{2}}+j/\sqrt{2} & -1 & \frac{1}{\sqrt{2}}-j/\sqrt{2} & j & -\frac{1}{\sqrt{2}}-j/\sqrt{2} \\
\checkmark 7 & 1 & j & -1 & -j & 1 & j & -1 & -j \\
8 & 1 & \frac{1}{\sqrt{2}}+j/\sqrt{2} & j & -\frac{1}{\sqrt{2}}+j/\sqrt{2} & -1 & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -j & 1/\sqrt{2}-j\,1/\sqrt{2}
\end{array}
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$
= \begin{bmatrix}
1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 1\cdot 0 + 1\cdot 0 \\
\cancel{1} + \frac{\cancel{1}}{\sqrt{2}} - j/\sqrt{2} - j - \frac{\cancel{1}}{\sqrt{2}} - \cancel{j}/\sqrt{2} - \cancel{1} - \frac{1}{\sqrt{2}} + \cancel{j}/\sqrt{2} + 0 + 0 \\
\cancel{1} + (-\cancel{j}) - \cancel{1} + \cancel{j} + 1 + (-j) + 0 + 0 \\
\cancel{1} + (-\frac{\cancel{1}}{\sqrt{2}} - \cancel{j}/\sqrt{2}) + j + \frac{\cancel{1}}{\sqrt{2}} - j/\sqrt{2} - \cancel{1} + \frac{1}{\sqrt{2}} + \cancel{j}/\sqrt{2} + 0 + 0 \\
\cancel{1 - 1 + 1 - 1 + 1 - 1} + 0 + 0 \\
\cancel{1} + (-\frac{\cancel{1}}{\sqrt{2}} + j/\sqrt{2}) - j + \frac{\cancel{1}}{\sqrt{2}} + \cancel{j}/\sqrt{2} - \cancel{1} + \frac{1}{\sqrt{2}} + \cancel{j}/\sqrt{2} + 0 + 0 \\
\cancel{1} + \cancel{j} - \cancel{1} - \cancel{j} + 1 + j + 0 + 0 \\
\cancel{1} + \frac{\cancel{1}}{\sqrt{2}} + j/\sqrt{2} + j - \frac{\cancel{1}}{\sqrt{2}} + \cancel{j}/\sqrt{2} - \cancel{1} - \frac{1}{\sqrt{2}} - \frac{\cancel{j}}{\sqrt{2}} + 0 + 0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
6 \\
-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} - j \\
1 - j \\
\frac{1}{\sqrt{2}} + j - j/\sqrt{2} \\
0 \\
\frac{1}{\sqrt{2}} - j + j/\sqrt{2} \\
1 + j \\
-\frac{1}{\sqrt{2}} + j + j/\sqrt{2}
\end{bmatrix}
$$

$$
X_8 = \left\{ 6,\ -\frac{1}{\sqrt{2}} - j\left(1 + \frac{1}{\sqrt{2}}\right),\ 1 - j,\ \frac{1}{\sqrt{2}} + j\left(1 - \frac{1}{\sqrt{2}}\right),\ 0,\ \frac{1}{\sqrt{2}} - j\left(1 - \frac{1}{\sqrt{2}}\right),\ 1 + j,\ -\frac{1}{\sqrt{2}} + j\left(1 + \frac{1}{\sqrt{2}}\right) \right\}
$$

Ans

§ compute 4-point IDFT of the sequence

$y(k) = \{1, 0, 1, 0\}$

on

Ans

$Y(k)$ or $X(k) = \{1, 0, 1, 0\}$

$\text{IDFT} = \frac{1}{N} W_N^* X_N$.

$$\cancel{x}\,y(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & \mp j \\ 1 & -1 & 1 & -1 \\ 1 & \mp j & -1 & +j \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 \\ 1-1 \\ 2 \\ 1-1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/4 \\ 0 \\ 2/4 \\ 0 \end{bmatrix} = \left\{ 1/2, 0, 1/2, 0 \right\}$$

Q compute 4-point DFT $x(n) = \{5, 6, 2, 1)$

Ans

$X_N = W_N \cdot x(n)$.

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+6+2+1 \\ 5+(-j\cdot 6)+(-2)+j \\ 5-6+2-1 \\ 5+j6-2-j \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ 3-5j \\ 0 \\ 3+5j \end{bmatrix}$$

$$X(K) = \{14, 3-5j, 0, 3+5j\}$$

~~Properties of DFT~~

~~4.5 compute DFT as a linear Transformation~~

4.6 Relationship of the DFT to other Transform.

- Relationship to the Fourier Series co-efficients of a periodic sequence.

A periodic sequence $\{x_p(n)\}$ with fundamental period N can be represented in a Fourier Series.

$$x_p = \sum_{K=0}^{N-1} C_K e^{j2\pi nK/N}, \quad -\infty < n < \infty$$

where the Fourier series co-efficients are given by

$$C_K = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nK/N}, \quad K = 0, 1, \ldots\ldots, N-1$$

So Simply.

$$\boxed{X(K) = N\, C_K}$$

* Relationship to the Fourier transform of an aperiodic sequence.

If $x(n)$ is an aperiodic finite energy sequence with Fourier transform $X(\omega)$, which is sampled at N equally spaced frequencies

$$\omega_k = \frac{2\pi k}{N}, \quad k = 0, 1, \ldots N-1;$$

$$X(k) \doteq X(\omega)\Big|_{\omega = 2\pi k/N} = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j2\pi n k/N}$$

$$k, 0, 1, \ldots N-1$$

are the DFT co-efficients of the periodic sequence of period N, given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

Thus, $x_p(n)$ is determined by aliasing $\{x(n)\}$ over the interval $0 \leq n \leq N-1$. The finite-duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise.} \end{cases}$$

$x(n)$ is finite duration & length $L \leq N$.

$$x(n) = \hat{x}(n) \quad 0 \leq n \leq N-1$$

* Relationship to the Z-Transform

let us consider a sequence $x(n)$ having the Z-Transform

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}$$

ROC that include the unit circle. If $X(Z)$ is sampled at the N equally spaced points on the unit circle $Z_k = e^{j2\pi k/N}$, $0, 1, 2, \ldots, N-1$

$$X(k) \equiv X(Z)\Big|_{Z = e^{j2\pi n k/N}}, \quad k = 0, 1, \ldots N-1$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi nk}{N}}$$

if the sequence $x(n)$ has finite duration of the ~~Ex~~ length N or less, the sequence can be recovered from it's N-point DFT. Hence it's Z-transform is uniquely determined by it's N-point DFT. $X(k)$

$$X(z) = \sum_{n=0}^{N-1} x(n) \cdot z^{-n}$$

$$= \sum_{k=0}^{N-1} \left[\frac{1}{N} \sum_{n=0}^{N-1} X(k) \cdot e^{j\frac{2\pi kn}{N}} \right] z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{-j\frac{2\pi kn}{N}} z^{-1} \right)^{n}$$

$$= \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{j2\pi kn/N} z^{-1}}$$

when evaluates on the unit circle. the fourier-Transform of the finite duration sequence. in terms of it' DFT.

$$X(\omega) = \frac{1-e^{j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{-j(\omega-2\pi k/N)}}$$

* Relationship to the fourier series co-efficients of a continuous - time signal.

Suppose that $x_a(t)$ is a continuous - time periodic signal with a fundamental period $T_p = 1/F_0$. The signal can be expressed in a fourier series

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{j2\pi k t F_0}$$

Where c_k are the fourier co-efficients.
If we sample $x_a(t)$ at a uniform rate $F_s = N/T_P = 1/T$, we obtain the discrete-time sequence

$$x(n) \equiv x_a(nT) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 n T}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kn/N}$$

$$= \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{j2\pi kn/N}$$

It is clear that

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N \tilde{c}_k$$

$$\boxed{\tilde{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN}}$$

Thus The $\{\tilde{c}_k\}$ sequence is an aliased version of the sequen $\{c_k\}$

4.7 Properties of the DFT

$$DFT = X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}, \quad k = 0, 1, \cdots N-1$$

$$IDFT = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}, \quad n = 0, 1, \cdots N-1$$

$$W_N = e^{-j2\pi/N}$$

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

Periodicity

If $x(n)$ & $X(k)$ are an N-point DFT pair then

$$x(n+N) = x(n) \text{ for all } n$$
$$X(K+N) = X(K) \text{ for all } K$$

These periodicities in $x(n)$ & $X(K)$ follow the immediately from formulas for DFT & IDFT.

Linearity

if $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

then for any real-valued or complex-valued constants a_1 & a_2.

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(K) + a_2 X_2(K)$$

Circular Symmetries of a Sequence

the N-point DFT of a finite duration sequence $x(n)$, of length $L \leq N$, is equivalent to the N-point DFT of a periodic sequence $x_p(n)$ of period N, which is obtained by the periodically extending $x(n)$:

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

Now Suppose that we shift the periodic sequence $x_p(n)$ by K units to the right.

$$x_p'(n) = x_p(n-K) = \sum_{l=-\infty}^{\infty} x(n-K-lN)$$

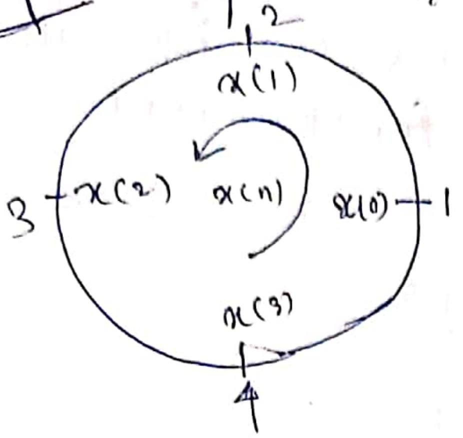
Ex ~~of~~ the Sequence given $x(n) = \{1, 2, 3, 4\}$ Circular shift ~~to~~ the Sequence by '2'.

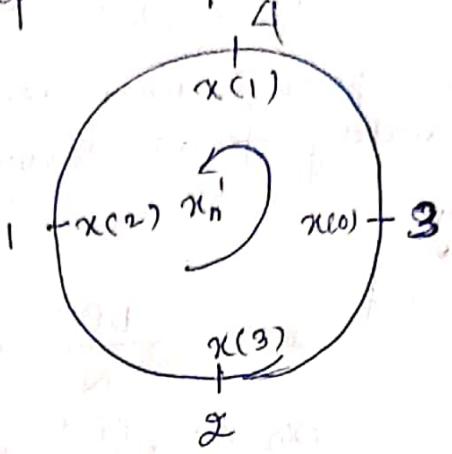
Ans

$k = 2$

$\underline{N = 4}$

~~$x_p(n) \to x_p'$~~ $x_p'(n) = x_p(n-2)$





* A N-point sequence is called Circulary even if it is symmetric

$$x(N-n) = x(n), \quad 1 \le n \le N-1$$

if odd

$$x(N-n) = -x(n), \quad 1 \le n \le N-1$$

for periodic sequence

even $x_p(n) = x_p(-n) = x_p(N-n)$

odd $x_p(n) = -x_p(-n) = -x_p(N-n)$

if the periodic sequence is complex valued, we have

conjugate even: $x_p(n) = x_p^*(N-n)$

conjugate odd: $x_p(n) = -x_p^*(N-n)$

$$x_p = x_{pe}(n) + x_{po}(n)$$

$$x_{pe}(n) = \frac{1}{2}\left[x_p(n) + x_p^*(N-n)\right]$$

$$x_{po}(n) = \frac{1}{2}\left[x_p(n) - x_p^*(N-n\right]$$

Symmetry properties of the DFT

$$x(n) = x_R(n) + j\,x_I(n) \quad 0 \le n \le N-1$$

$$X(k) = X_R(k) + j\,X_I(k) \quad 0 \le k \le N-1$$

Substituting expression of DFT

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi kn}{N} + x_I(n) \sin \frac{2\pi kn}{N} \right]$$

$$X_I(k) = - \sum_{n=0}^{N-1} \left[x_R(n) \sin \frac{2\pi kn}{N} - x_I(n) \cos \frac{2\pi kn}{N} \right]$$

IDFT

$$x_R(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \cos \frac{2\pi kn}{N} - X_I(k) \sin \frac{2\pi kn}{N} \right]$$

$$x_I(k) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \sin \frac{2\pi kn}{N} + X_I(k) \cos \frac{2\pi kn}{N} \right]$$

Real & even Sequen

$$x(n) = x(N-n), \quad 0 \le n \le N-1$$

$X_I(k) = 0$ Hence the DFT reduce to.

$$X(k) = \sum_{n=0}^{N-1} x(n) \cos \frac{2\pi kn}{N}, \quad 0 \le k \le N-1$$

IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cos \frac{2\pi kn}{N}, \quad 0 \le n \le N-1$$

Real & odd Sequences

$$x(n) = -x(N-n), \quad 0 \le n \le N-1$$

$$X_R(k) = 0$$

DFT

$$X(k) = -j \sum_{n=0}^{N-1} x(n) \sin \frac{2\pi kn}{N}, \quad 0 \le n \le N-1$$

IDFT

$$x(n) = j \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sin \frac{2\pi kn}{N}, \quad 0 \le k \le N-1$$

4.8 Multiplication of Two DFTs & circular convolution

Sol

We have two finite-duration sequence of length N, $x_1(n)$ & $x_2(n)$

N-point DFT

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N}, \quad k=0,1,\ldots N-1$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) \cdot e^{-j2\pi nk/N}, \quad k=0,1,\ldots N-1$$

we have

$$X_3(k) = X_1(k) \cdot X_2(k), \quad k=0,1,\ldots N-1$$

IDFT

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) \cdot e^{j2\pi km/N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_2(k) \cdot e^{j2\pi km/N}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{-j2\pi k(m-n-l)/N} \right]$$

we finally obtain the

$$\boxed{x_3(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m-n)_N}$$

Question

Perform the circular convolution of the following two sequence

$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

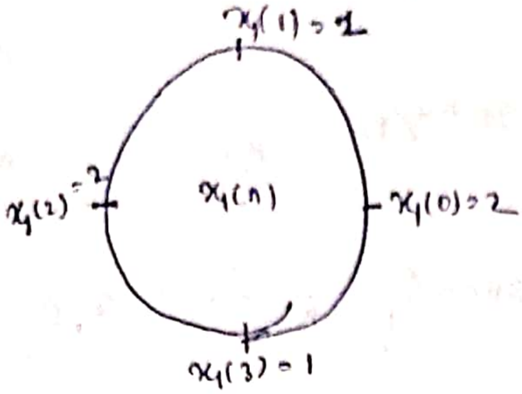
$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

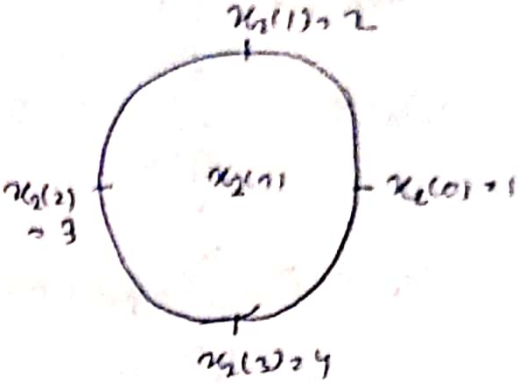
Ans

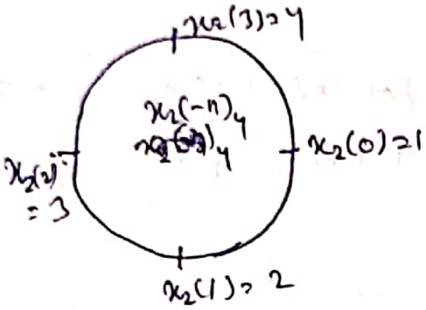
$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m-n)_N$$

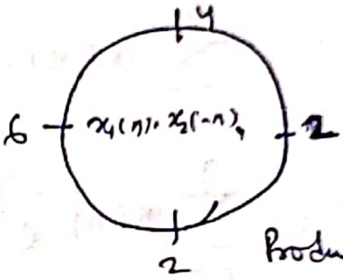
Beginning with m=0

$$x_3(0) = \sum_{n=0}^{3} x_1(n) \cdot x_2(-n)_N$$



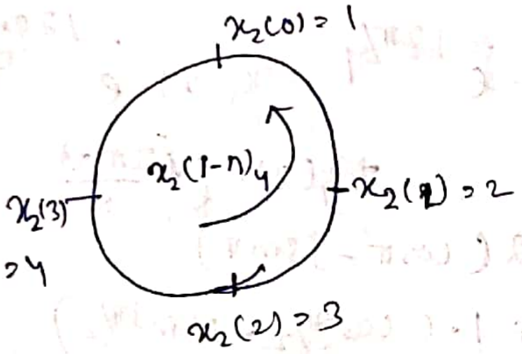


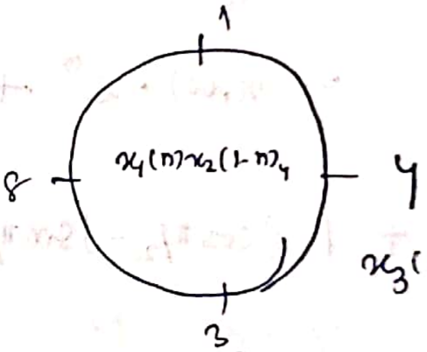




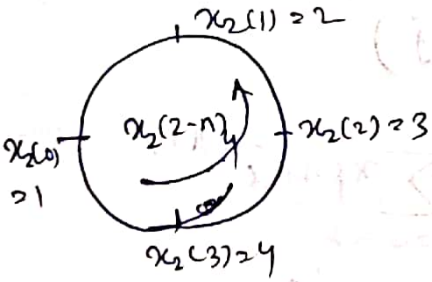
$x_3(0) = 14$

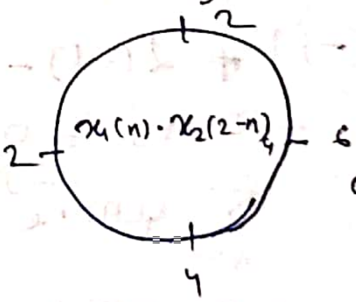
Product Seque



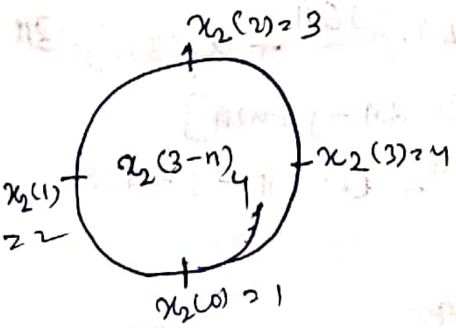


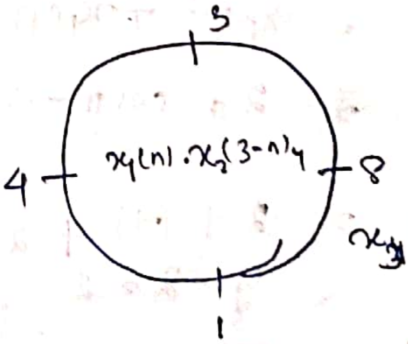
$x_3(1) = 16$





$x_3(2) = 14$





$x_3(3) = 16$

$x_3(n) = \{14, 16, 14, 16\}$

Q By means of the DFT & IDFT, determine the sequence corresponding circular convolution of

$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$, $x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

Ans

First we DFT

$$X_1(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi nk/N} \quad . \ k = 0, 1, \cdots N-1$$

$$X_1(k) = \sum_{n=0}^{3} x(n) \cdot e^{-j2\pi nk/4} \quad k, = 0, 1, 2, 3$$

$$X_1(0) = \sum_{n=0}^{3} x(n) \cdot e^{0} = x_1(0) + x_1(1) + x_1(2) + x_1(3)$$
$$= 2 + 1 + 2 + 1 = 6$$

$$X_1(1) = \sum_{n=0}^{3} x(n) \, e^{-j\frac{2\pi n}{4}}$$

$$= x_1(0) \cdot e^{0} + x_1(1) \cdot e^{-j2\pi/4} + x_1(2) \cdot e^{-j\frac{2\pi \cdot 2}{4}} + x_1(3) \cdot e^{-j\frac{2\pi \cdot 3}{4}}$$

$$= 2 + 1 \cdot (\cos \pi/2 - j \sin \pi/2) + 2(\cos \pi - j \sin \pi) + 1 \cdot (\cos 3\pi/2 - j \sin 3\pi/2)$$

$$= 2 + (-j) + 2(-1) - 0 + (0 + j)$$

$$= 2 - j - 2 + j = 0$$

$$X_1(2) = \sum_{n=0}^{3} x_1(n) \cdot e^{-j\frac{2\pi \cdot 2n}{4}} = \sum_{n=0}^{3} x(n) \cdot e^{-j\pi n}$$

$$= x_1(0) + x_1(1) \cdot e^{-j\pi} + x_1(2) \cdot e^{-j2\pi} + x_1(3) \cdot e^{-j3\pi}$$

$$= 2 + \cos \pi - j \sin \pi + 2 \cdot [\cos 2\pi - j \sin 2\pi] + \cos 3\pi - j \sin 3\pi$$

$$= 2 + (-1) + 2 \cdot 1 + -1$$

$$= 4 - 2 = 2$$

$$X_1(3) = \sum_{n=0}^{3} x_1(n) \cdot e^{-j\frac{2\pi \cdot 3n}{4}} = \sum_{n=0}^{3} x_1(n) \cdot e^{-j\frac{3\pi n}{2}}$$

$$= x_1(0) + x_1(1) \cdot e^{-j3\pi/2} + x_1(2) \cdot e^{-j3\pi} + x_1(3) \cdot e^{-j\frac{9\pi}{2}}$$

and $x_1(3) = 0$

DFT of $x_2(n)$

$$X_2(k) = \sum_{n=0}^{3} x_2(n) \cdot e^{-\frac{j2\pi nk}{4}}$$

$1 + 2e^{j\pi k/2} + 3e^{-j\pi k} + 4e^{-j3\pi k/2}$

$x_2(0) = 1 + 2 + 3 + 4 = 10$

$x_2(1) = 1 + 2 \cdot e^{-j\pi/2} + 3 \cdot e^{-j\pi} + 4e^{-j3\pi/2} = -2 + j2$

$x_2(2) = 1 + 2.(\cos\pi/2 - j\sin\pi/2)$ ~~$+3(\cos\pi - j \sin\pi)$~~

$+4.($~~$\cos 3\pi/2 - j\sin 3\pi/2$~~$)$

$=$ ~~$1 + (2j) + 3(-1) + 4j$~~

$=$ ~~[illegible] $2j$~~ -2

$=$

$x_2(3) = 1 + 2 \cdot e^{-j\pi 3/2} + 3 \cdot e^{-j\pi 3} + 4 e^{-9\pi/2}$

$=$ ~~$1 + 2[j] + 3[-1] +$~~ $4 \quad -2 - j2$

$=$ ~~$1 + 2j - 3$~~ $+ 4$

$=$ ~~[illegible]~~

$X_3(k) = X_1(k) \cdot X_2(k)$

$X_3(0) = 60, \quad X_3(1) = 0 \quad X_3(2) = -4, \quad X_3(3) = 0$

IDFT

$$x_3(n) = \frac{1}{4}\sum_{k=0}^{3} X_3(k) \cdot e^{j2\pi nk/4} \qquad n = 0, 1, 2, 3$$

$$= \frac{1}{4}(60 - 4e^{j2\pi n j})$$

$x_3(0) = 14, \quad x_3(1) = 16, \quad x_3(2) = 14, \quad x_3(3) = 16$

Circular convolution

$$x_1(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(k)$$

$$x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_2(k)$$

$$x_1(n) \text{ Ⓝ } x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(k) \cdot X_2(k)$$

Q Determine the sequence y(n) that result use of 4- point DFT of sequen h(n) = {1, 2, 3}
x(n) = {1, 2, 2, 1}

Ans

DFT → H(n)

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$H_4(n) = W_4 \cdot h_4$

$$H_4(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3 \\ 1-2j-3+0 \\ 1-2+3-0 \\ 1+2j-3+0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2-2j \\ 0 \\ -2+j2 \end{bmatrix}$$

DFT of x(n) = $X_4(K) = W_4(n) \cdot x_4(n)$

$$x_4(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$H_4(K) = \begin{bmatrix} 6 \\ 1-2j-2+j \\ 1-2+2-1 \\ 1+2j-2j-j \end{bmatrix} = \begin{bmatrix} 6 \\ -1-j \\ 0 \\ -1+j \end{bmatrix}$$

Product = $H_4(K) \cdot X_4(K)$

$Y(0) = 36$, $Y(1) = j4$, $Y(2) = 0$ $Y(3) = -j4$

Q- Find

seque

using T

Ans

$x_2(2) = 3$

$x_2(2) = 1$

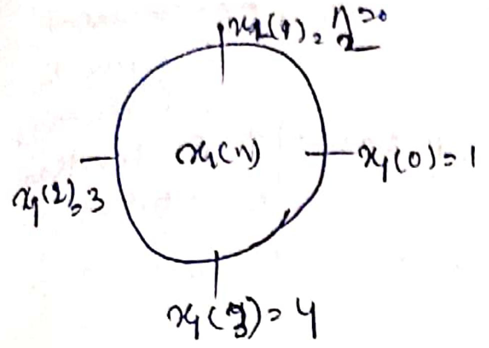
$x_2(3) = 2$

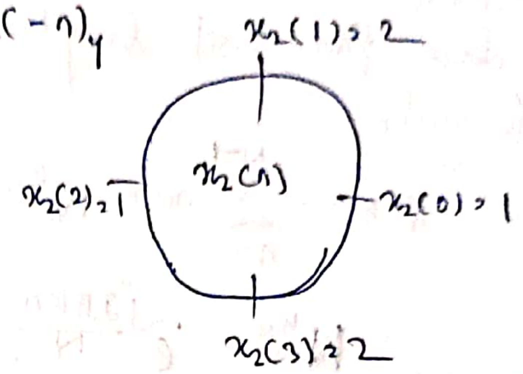
Q. Find circular convolution and for the following sequence $x_1(n) = \{1, 2, 3, 4\}$ $x_2(n) = \{1, 2, 1, 2\}$ using Time Domain formula method.

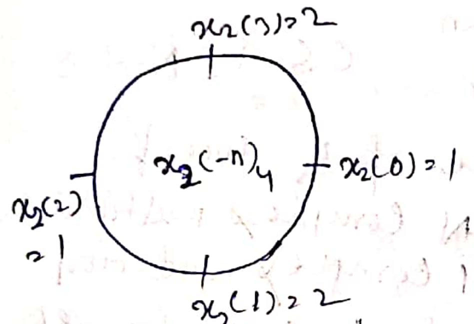
Ans
$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m-n)_N$$

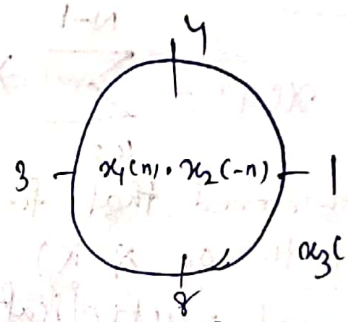
L = 4 M = 4

$$x_3(0) = \sum_{n=0}^{3} x_1(n) \cdot x_2(-n)_4$$

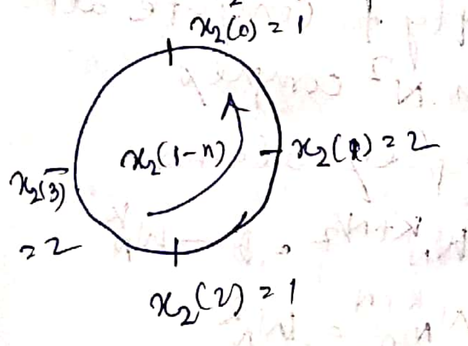


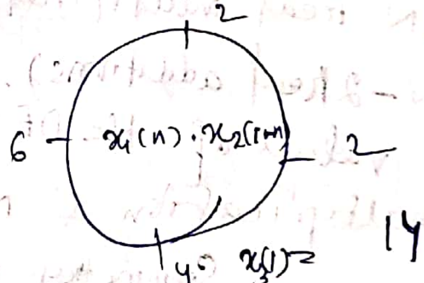




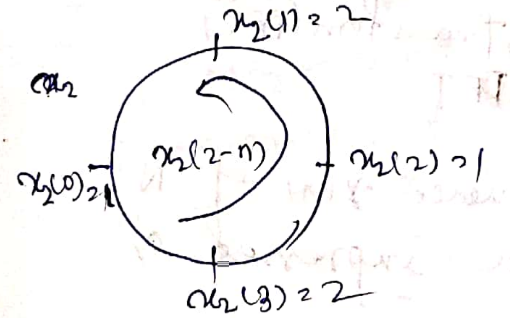


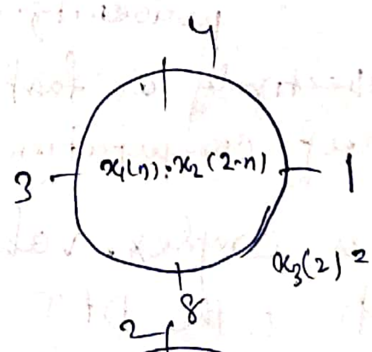
$x_3(0) = 16$



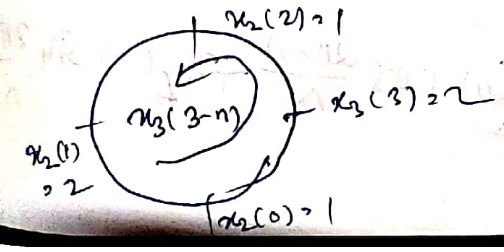


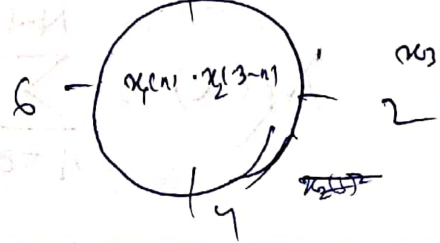
$x_3(1) = 14$





$x_3(2) = 16$





$x_3(3) = 14$

CH-5 Fast Fourier Transform Algorithm & Digital Filter

5.1 Efficient computation of the DFT: FFT algorithms

→ DFT in various digital signal processing application such as linear filtering, correlation analysis, & Spectrum analysis.

DFT is to compute the sequence $\{X(k)\}$ of N complex-valued numbers given another sequence of data $\{x(n)\}$ of length N, according to the formula

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad 0 \le K \le N-1$$

→ FFT consumes less time & calculation.
→ faster than DFT

$$W_N^{kn} = e^{\frac{-j2\pi kn}{N}}.$$

IDFT

$$x(n) = \sum_{k=0}^{N-1} X(K) \cdot W_N^{-kn} \qquad 0 \le n \le N-1$$

we observed that for each value of K direct computation of X(K) involves N complex multiplication (4N real multiplication) & N-1 complex addition (4N-2 real additions). consequently to compute all N values of the DFT requires N^2 complex multiplication & N^2-N complex addition.

Symmetry property $= W_N^{K+N/2} = -W_N^K$

periodicity property $= W_N^{k+N} = W_N^K$

Collectively as fast fourier Transform (FFT)

5.2 Direct computation of the DFT

For a complex-valued sequence x(n) of N points, the DFT may be expressed as

$$X_R(K) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi kn}{N} + x_I(n) \sin \frac{2\pi kn}{N} \right]$$

$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi kn}{N} - x_I(n)\cos\frac{2\pi kn}{N}\right]$$

The direct computation of requir

1. $2N^2$ evaluation of trigonometric function.
2. $4N^2$ real multiplication.
3. $4N(N-1)$ real addition.
4. A number of indexing & addressing operation.

5.3 Radix – 2 FFT algorithm

Four algorithms for efficient computation of the DFT based on the divided – and – conquer approach. the number N of data points is not a prime. the approach is very efficient. When N is highly composite. when N can be factored as

$$N = r_1 r_2 r_3 \cdots r_v$$ where the $[r_j]$

are prime.

as case $r_1 = r_2 = \cdots = r_v \equiv r$ So

$$N = r^v$$

Case of DFT are of size r, so that computation of N-point DFT has a regular pattern. The number r is called the radix of the FFT algorithm.

Radix-2 algorithm widely used :

Let $N = 2^v$ point DFT by dived - conquer we select $M = N/2$ & $L=2$. This selection results in a split of the N-point data sequence into two N/2 - point data sequence $f_1(n)$ & $f_2(n)$ corresponding to the even - numbered & odd numbered sample of $x(n)$

$$f_1(n) = x(2n)$$
$$f_2(n) = x(2n+1), \quad n = 0, 1, \ldots \frac{N}{2}-1$$

Now N - point DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}, \quad k = 0, 1, \dots N-1$$

$$= \sum_{n=\text{even}} x(n) W_N^{kn} + \sum_{n=\text{odd}} x(n) W_N^{kn}$$

$$= \sum_{m=0}^{N/2-1} x(2m) \cdot W_N^{2mk} + \sum_{m=0}^{N/2-1} x(2m+1) W_N^{k(2m+1)}$$

We substitute $W_N^2 = W_{N/2}$

$$= \sum_{m=0}^{N/2-1} f_1(m) W_N^{2mk} + \sum_{m=0}^{N/2-1} f_2(m) W_{N/2}^{km}$$

$$= F_1(k) + W_N^k F_2(k), \quad k = 0, 1, 2 \dots N-1$$

$$X(k) = F_1(k) + W_N^k F_2(k), \quad k = 0, 1, 2, \dots N/2-1$$

$$X(k + N/2) = F_1(k) + W_N^k F_2(k), \quad k = 0, 1, 2, \dots N/2-1$$

if $G_1(k) = F_1(k)$, $k = 0, 1, \dots \frac{N}{2}-1$

$G_2(k) = W_N^k F_2(k)$, $k = 0, 1, \dots N/2-1$

Then the DFT $X(k)$ may be express

$$X(k) = G_1(k) + G_2(k), \quad k = 0, 1, \dots N/2-1$$

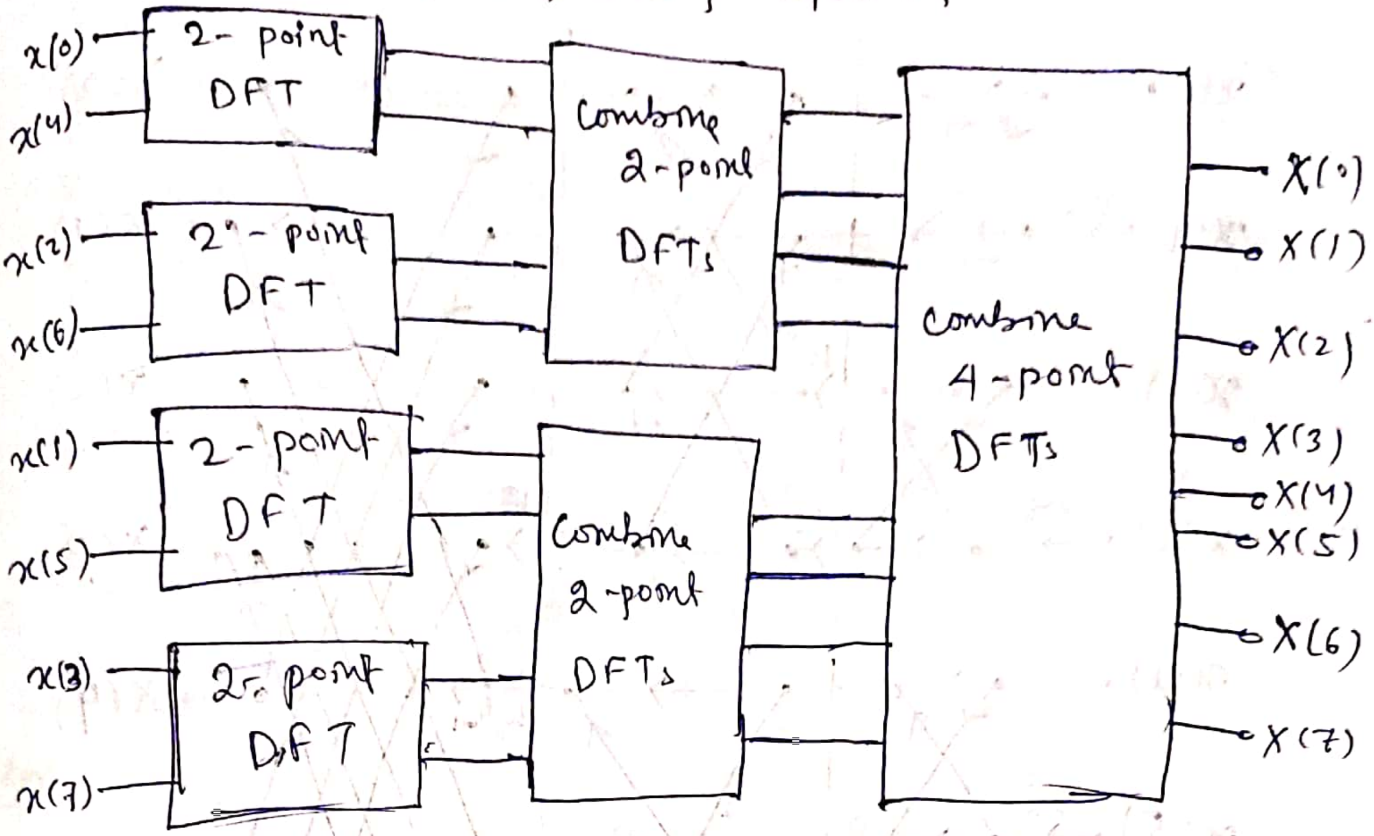
$$X(k + \tfrac{N}{2}) = G_1(k) - G_2(k), \quad k = 0, 1, \dots N/2-1$$

8- point DIT FFT flowgraph.

8pt; $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

4pt; $\left\{\begin{array}{l} x_e(n) = \{x(0), x(2), x(4), x(6)\} \\ x_o(n) = \{x(1), x(3), x(5), x(7)\} \end{array}\right.$

8pt. $x_1(n) = \{x(0), x(4)\}$ $x_3(n) = \{x(1), ~~[illegible]~~ x(5)\}$
$x_2(n) = \{x(2), x(6)\}$ $x_4(n) = \{x(3), x(7)\}$

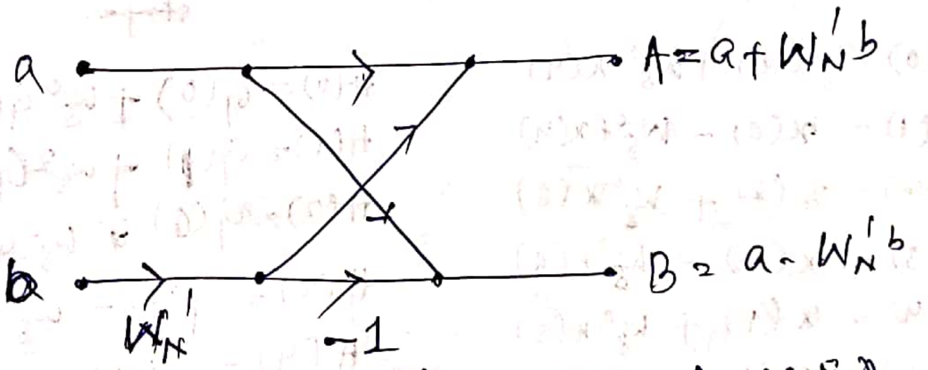


(Three stages in the computation of an N = 8 point DFT.)

* DIT-FFT algorithm.

DIT - Decimation - in - time FFT

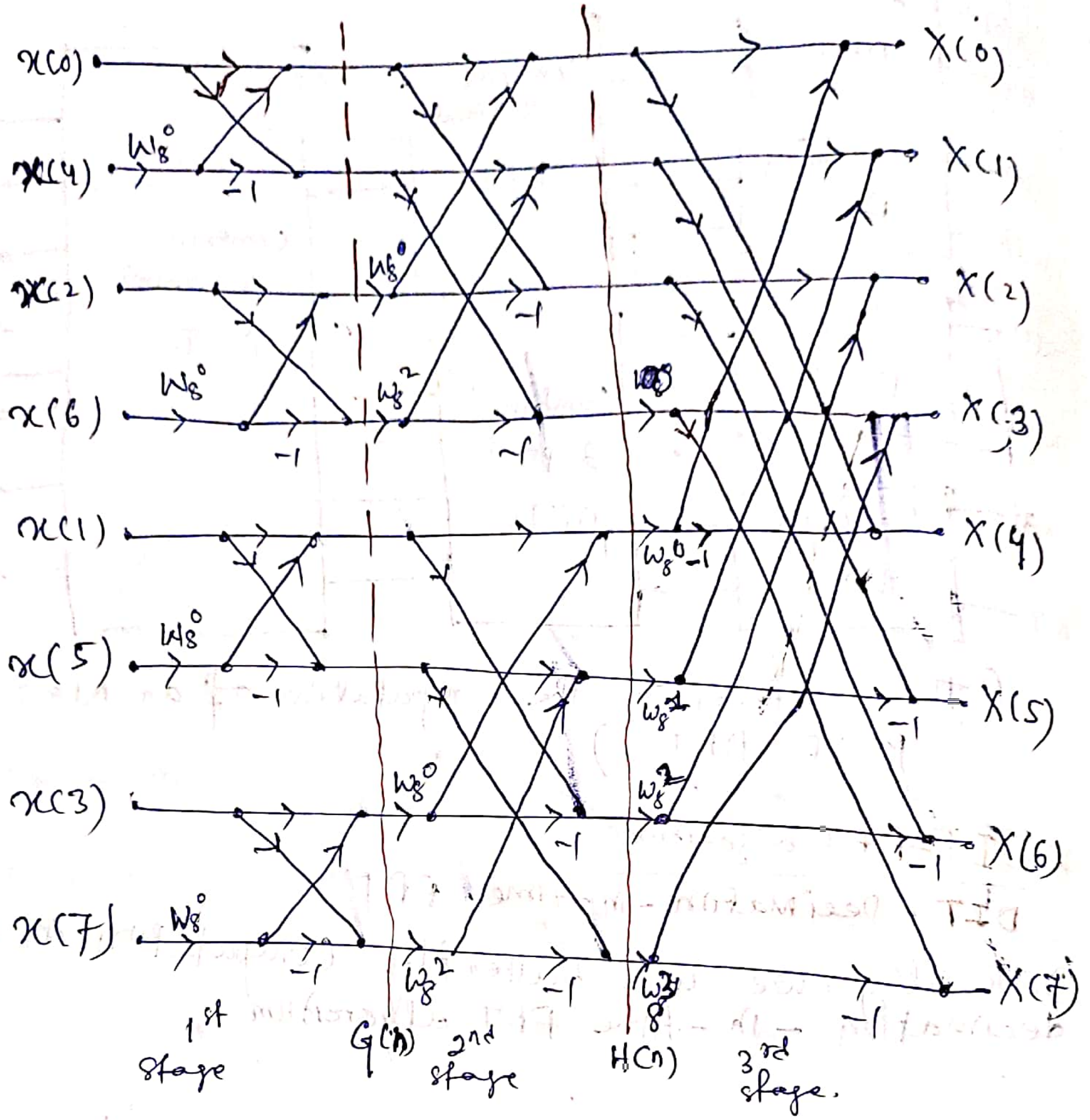
Basically we use butterfly computation in the decimation - in - time FFT algorithm.



Given sequence x(n) then we get X(k)

In question x(n) is given we found X(k) = ?

* 8-point DIT FFT algorithm



$G(0) = x(0) + W_8^0\, x(4)$

$G(1) = x(0) - W_8^0\, x(4)$

$G(2) = x(2) + W_8^0\, x(6)$

$G(3) = x(2) - W_8^0\, x(6)$

$G(4) = x(1) + W_8^0\, x(5)$

$G(5) = x(1) - W_8^0\, x(5)$

$G(6) = x(3) + W_8^0\, x(7)$

$G(7) = x(3) \cancel{+} - W_8^0\, x(7)$

We know, $W_N^{kn} = e^{-\frac{j2\pi kn}{N}}$

$W_8^0 = e^0 = 1$

$H(0) = G(0) + W_8^0\, G(2)$

$H(1) = G(1) + W_8^2\, G(3)$

$H(2) = G(0) - W_8^0\, G(2)$

$H(3) = G(1) - W_8^2\, (G3)$

$H(4) = G(4) + W_8^0\, G(6)$

$H(5) = G(5) + W_8^2\, G(7)$

$H(6) = G(4) - W_8^0\, G(6)$

$H(7) = G(5) - W_8^2\, G(7)$

$X(0) = H(0) + W_8^0 H(4)$

$X(1) = H(1) + W_8^1 H(5)$

$X(2) = H(2) + W_8^2 H(6)$

$X(3) = H(3) + W_8^3 H(7)$

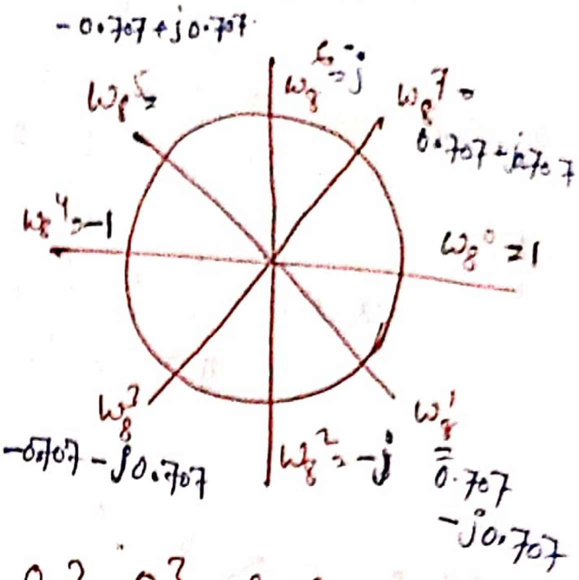
$X(4) = H(0) - W_8^0 H(4)$

$X(5) = H(1) - W_8^1 H(5)$

$X(6) = H(2) - W_8^2 H(6)$

$X(7) = H(3) - W_8^3 H(7)$

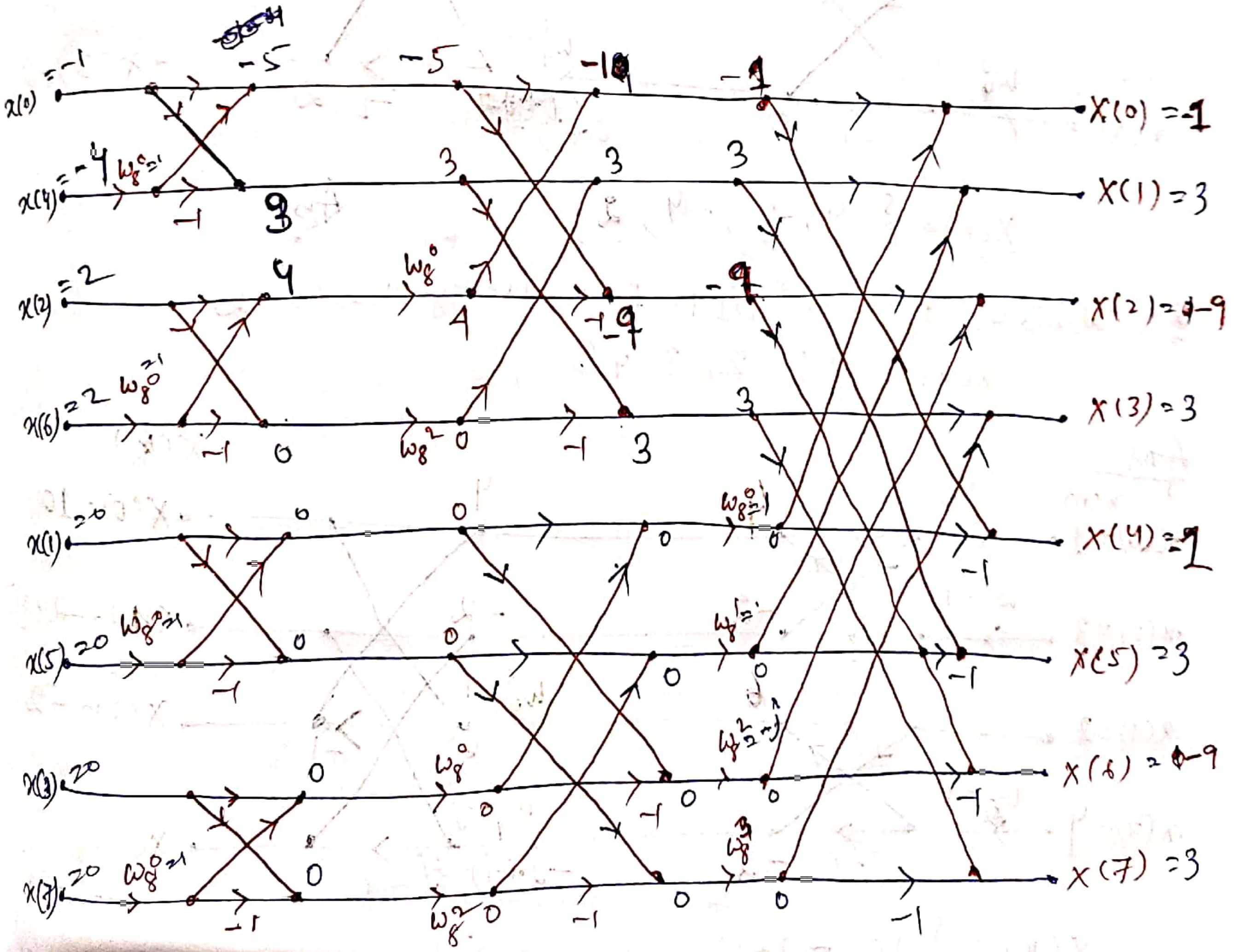
Note point



Q. $x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$ find Radix-2 DIT FFT ?

Ans

$x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$
$n = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$



$X(0) = -1$

$X(1) = 3$

$X(2) = -9$

$X(3) = 3$

$X(4) = 1$

$X(5) = 3$

$X(6) = -9$

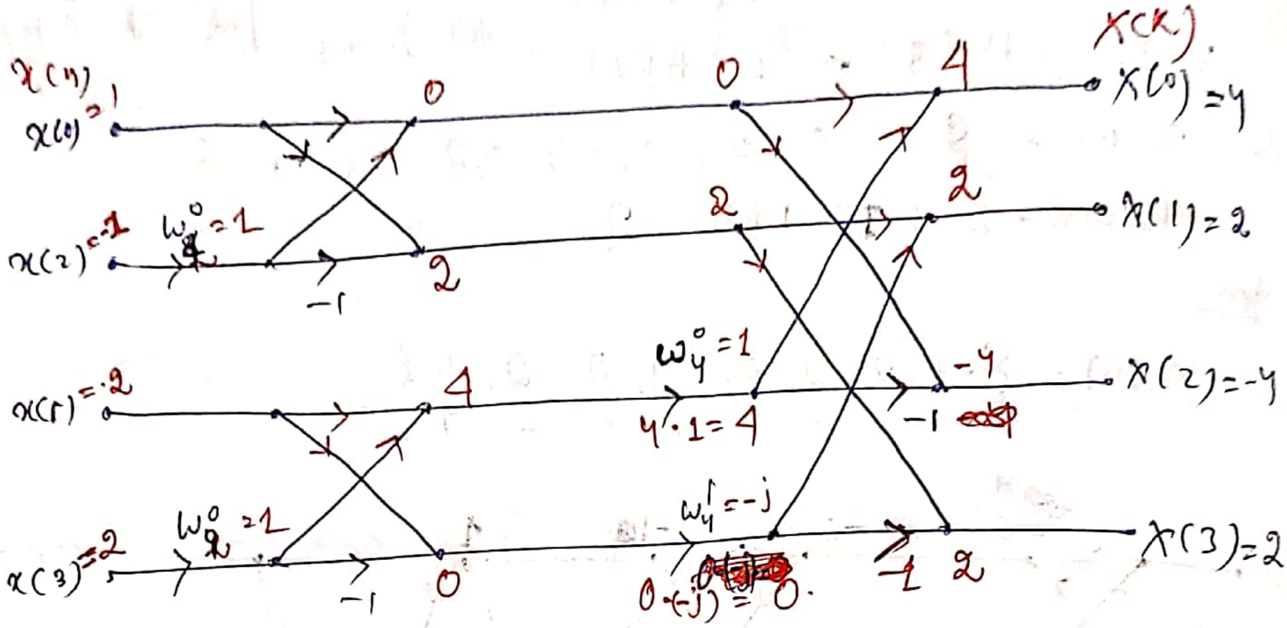
$X(7) = 3$

$X(K) = \{\underset{\uparrow}{-1}, 3, -9, 3, -1, 3, -9, 3\}$ Ans

Q Calculate 4-point DIT FFT using Radix-2
$x(n) = \{1, 2, -1, 2\}$

Ans

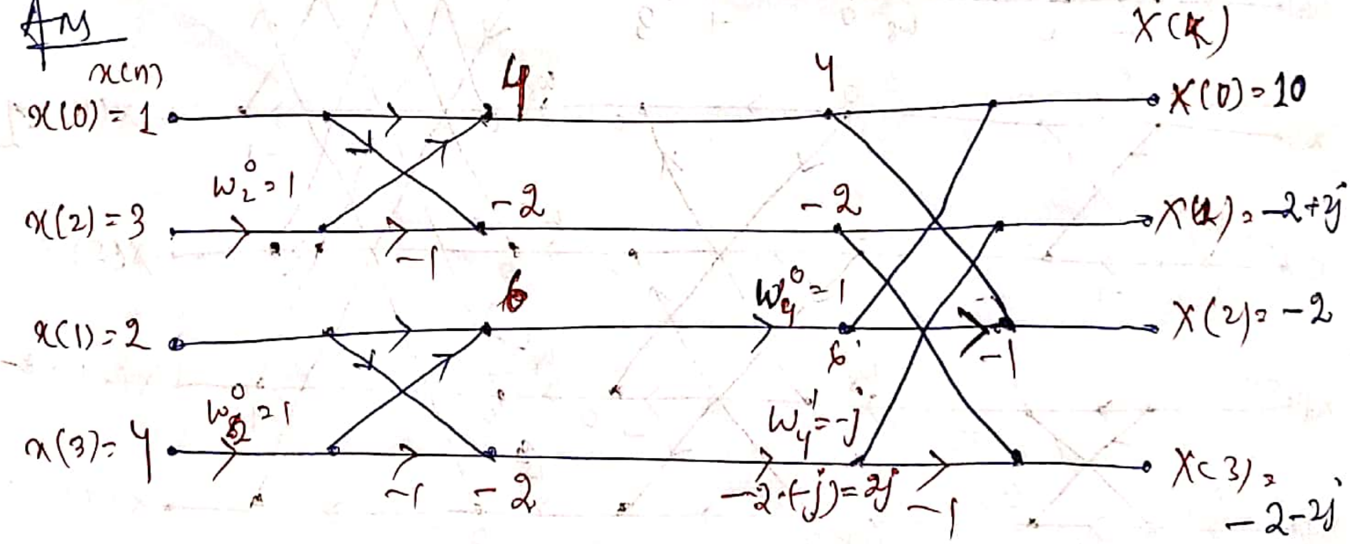
$x(n) = \{1, 2, -1, 2\}$
$n = 0, 1, 2, 3$



$X(k) = \{4, 2, -4, 2\}$ Ans

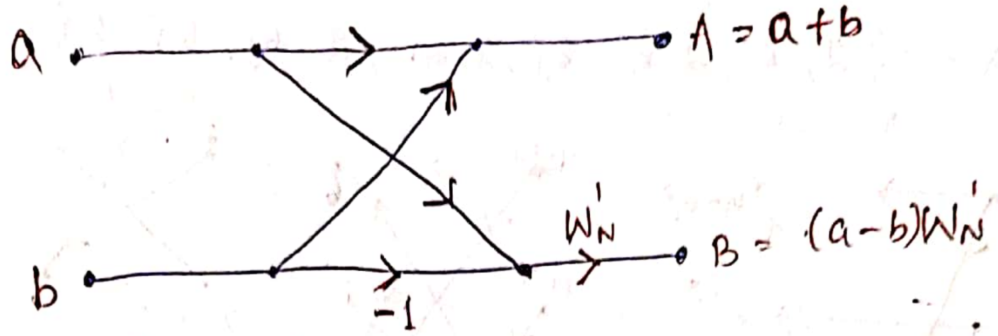
Q Calculate 4-point DIT-FFT using Radix-2
$x(n) = \{1, 2, 3, 4\}$

Ans

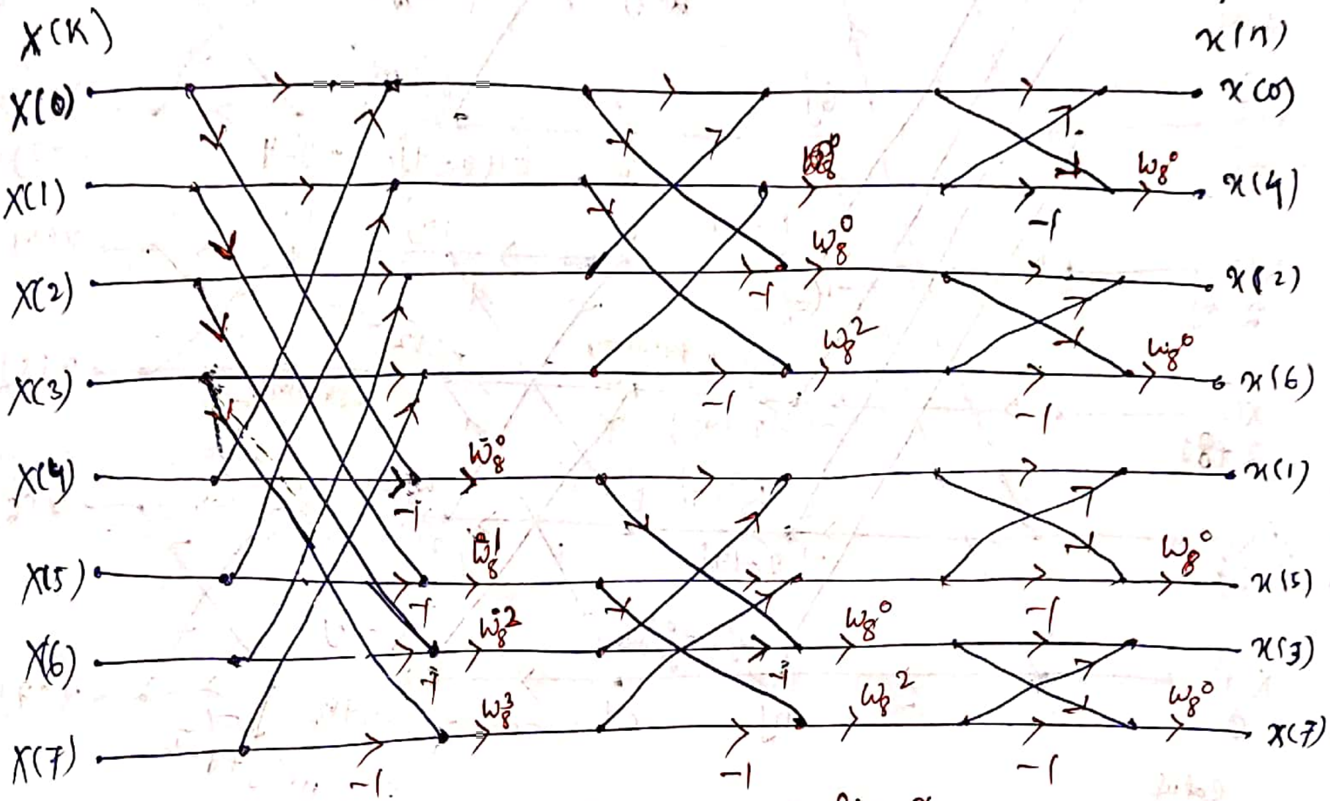


$X(k) = \{10, -2+2j, -2, -2-2j\}$

Decimation in frequency (DIF)-FFT Radix-2.



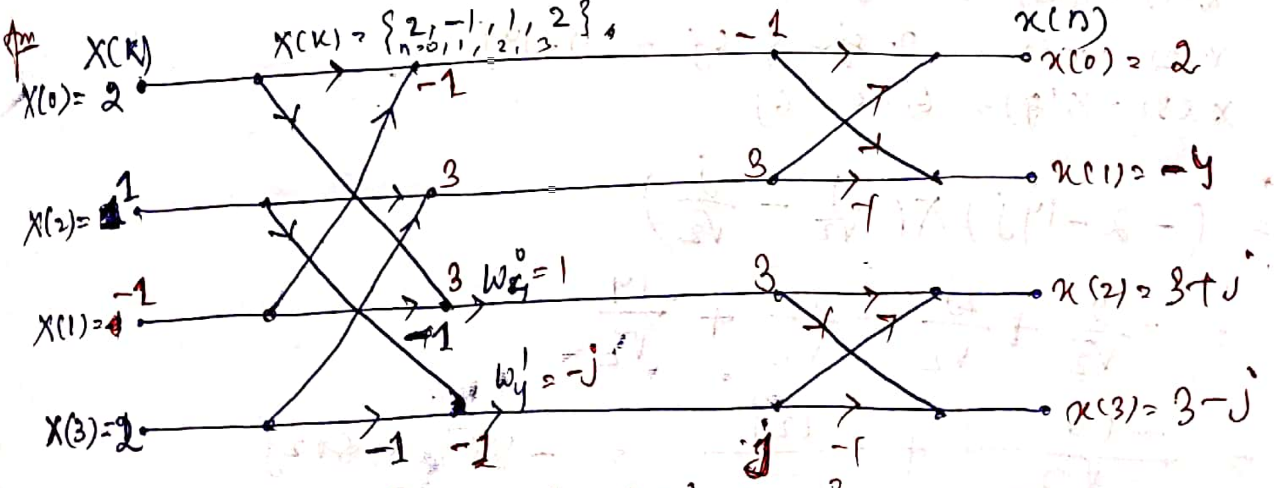
N=8 point DIF-FFT algorithm



Q calculate 4-point DIF-FFT using Radix-2

$X(K) = \{2, -1, 1, 2\}$

Ans



$x(n) = \{2, -4, 3+j, 3-j\}$

Q Compute DIF-FFT 8-point

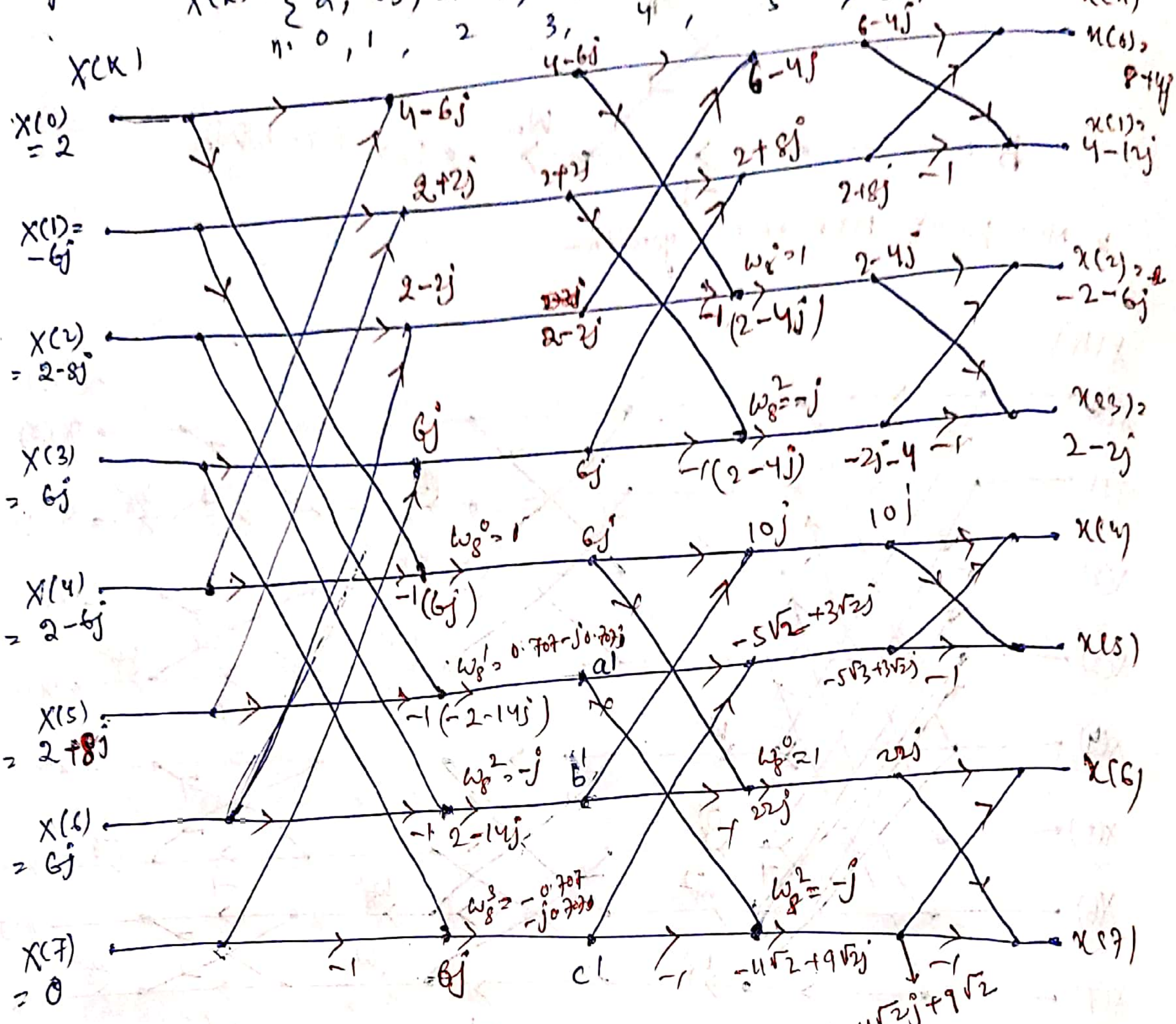
$X(K) = \{2, -6j, 2-8j, 6j, 2-6j, 2+8j, 6j, 0\}$

$X(K) = \{2, -6j, 2-8j, 6j, 2-6j, 2+6j, 6j, 0\}$.

Ans

$x(k) = \{2, -6j, 2-8j, 6j, 2-6j, 2+6j, 6j, 0\}$

n: 0, 1, 2, 3, 4, 5, 6, 7



calcu

$X(0) - X(4) = 2 - \{2-6j = 2-2+6j = 6j$

$X(1) - X(5) = -6j - \{2+8j) = -6j - 2 - 8j = -2-14j$

$X(2) - X(6) = 2-8j - \{6j = 2-14j$

$X(3) = X(7) = 6j - 0 = 6j$

$$a'^{*} = (-2-14j) \times \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)$$

$$= -\frac{2}{\sqrt{2}} + \frac{2j}{\sqrt{2}} - \frac{14j}{\sqrt{2}} + \frac{-14}{\sqrt{2}}$$

$$= \frac{-16}{\sqrt{2}} + \frac{12j}{\sqrt{2}} = -8\sqrt{2} + 6\sqrt{2}j$$

$b' = (2-14j) \times -j = -2j - 14j$

$= -16j$

$c' = 6j \times \left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} j\right]$

$= -\frac{6j}{\sqrt{2}} - \frac{6j^2}{\sqrt{2}} = -\frac{6j}{\sqrt{2}} + \frac{6}{\sqrt{2}} = 3\sqrt{2} - 3\sqrt{2}j$

$\times \; 4-6j - (2-2j)$ $\qquad \times \; 2+2j - 6j$

$\qquad\qquad\qquad\qquad\quad = 2-4j$

$4-6j-2+2j$

$= 2-4j$

$= -8\sqrt{2} + 6\sqrt{2}j - 3\sqrt{2} + 3\sqrt{2}j$

$* \; -11\sqrt{2} + 9\sqrt{2}j$

$=$

$x(1) = 6-4j - 2-8j = -4-12j$

$x(2) = 2-4j - 2j - 4 = -2-6j$

$x(3) = 2-4j - (-4-2j) = 2-4j+4+2j$

$= 2-2j$

$x(4) = -5\sqrt{3} + 3\sqrt{3}j + 10j$

$x(5) = 10j + 5\sqrt{3} - 3\sqrt{3}j$

$X(6) = -11\sqrt{2} + 9\sqrt{2}j + 22j$

$x(7) = 22j - 11\sqrt{2}j - 9\sqrt{2}$

5.5 Introduction to Digital filter (FIR filter)

Introduction

* Filters are two types ① analog filter & digital filter.

So we know

Fourier Transfor

$h(t) \xleftrightarrow{CTFT} H(j\omega)$

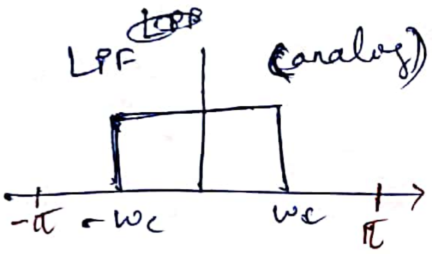
CTFT = continuous time fourier Transfor

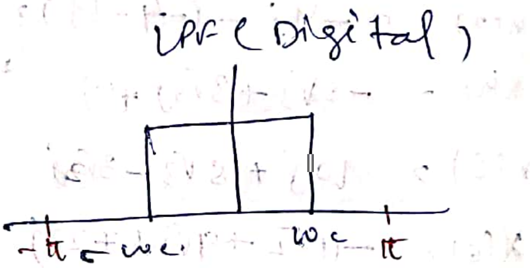
$h(n) \xleftrightarrow{DTFT} H(e^{j\omega})$

DTFT = Discrete Time fourier Transfor

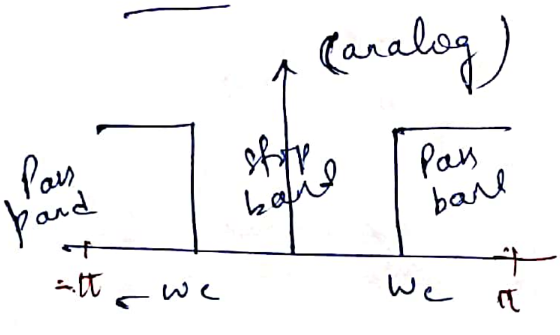
So analog filter we already know i.e. Low pass filter, High pass filter, band pass filter & band stop filter.

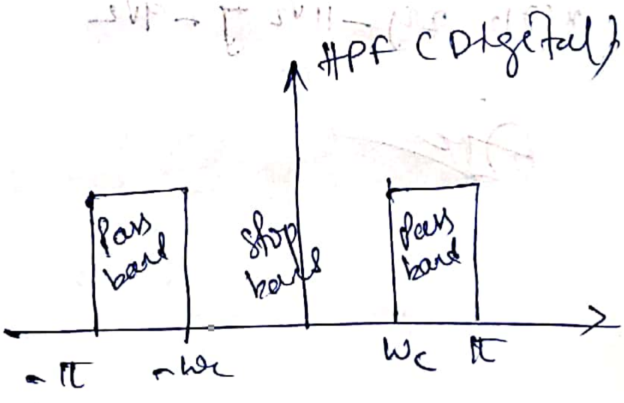
So we see





HPF





So we check at

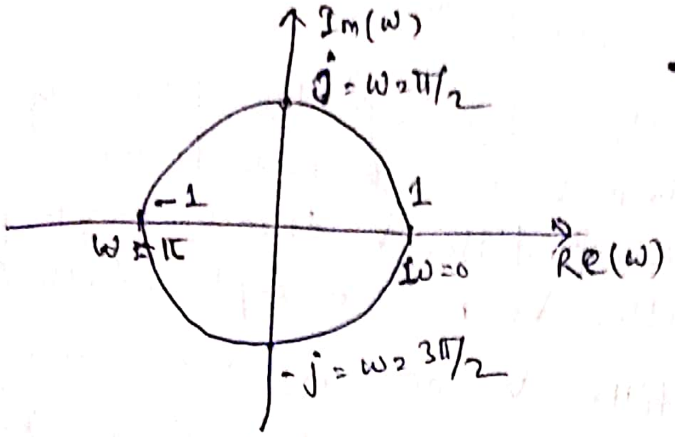
	low freq	~~low freq~~ High freq
Analog	$\omega=0$	$\omega=\infty$
Digital	$\omega=0$	$\omega=?$

So we have know what is the angular

frequency in Digital filter·

'So we already study Z - Transfer·

we know
$|Z| = re^{jw}$, r is magnitude



if $|z| = r = 1$

$\boxed{Z = e^{jw}}$

if we put $w = 0$

$Z = e^{0}$
$= 1$

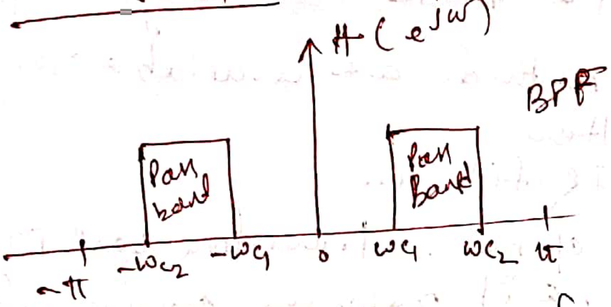
then

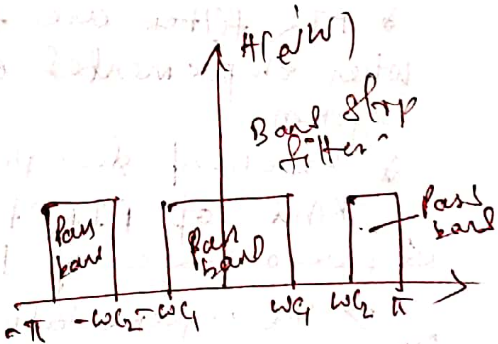
→ $\omega = \pi$:
$|Z| = e^{j\pi} = -1$

→ $W = \pi/2$
$|Z| = e^{j\pi/2} = j$

we already Z - Transform is discrete time function. So we study 'ω' value by putting different value

In digital





→ Electronic circuits which perform signal processing functions, remove unwanted frequency component, to enhance wanted ones both.

→ Digital filter - performs mathematical operation on a sampled, discrete time signal to reduce or enhance certain aspects of that signal.

→ filter generally do not add frequency component to a signal.

→ Boost or attenuate selected frequency region.

Classification of digital filter

→ Depending on the form of the filter eq^n linear filter.

a) Linear filter Vs non- linear filter Vs time-varying filter

b) Time - invariant filter Vs non - adaptive filter

c) Adaptive filter Vs non - recursive filter

d) Recursive filter Vs non - recursive filter

→ Depending on the structure of implementation

a) Direct form, cascade - form, parallel form, & lattice structure.

* Finite Impulse response filter :-

→ FIR filter are prefered over their IIR (Infinite Impulse response filter)

* FIR ~~always~~ filter always stable.

* FIR filter with exactly linear phase can easily be designed.

* FIR filter can be realized in both recursive & non - recursive structures.

* FIR filter are free of limit cycle oscillations, when implemented on a finite word length digital system:

* Excellent design methods are available for kinds of FIR filters.

disadvantages of FIR filter are

→ The implementation of narrow transition band FIR filters are very costly, as it requires considerably more arithmetic operation and hardware components such as multiplier, adders & delay elements

→ memory requirement & execution time are very high it operates only on previous values of the input

$$\boxed{y[n] = \sum_{k=0}^{M} b(k)\, x(n-k)}$$

5.5 Introduction to DSP architecture, familarsation of different types of processor

Different Types of processor

→ The programmable digital signal processors are general purpose microprocessor designed specifically for digital signal processing application. They contain special purpose architecture and instruction set so as to execute computation - intensive DSP algorithms more efficiently.

i) General purpose digital signal processor :-

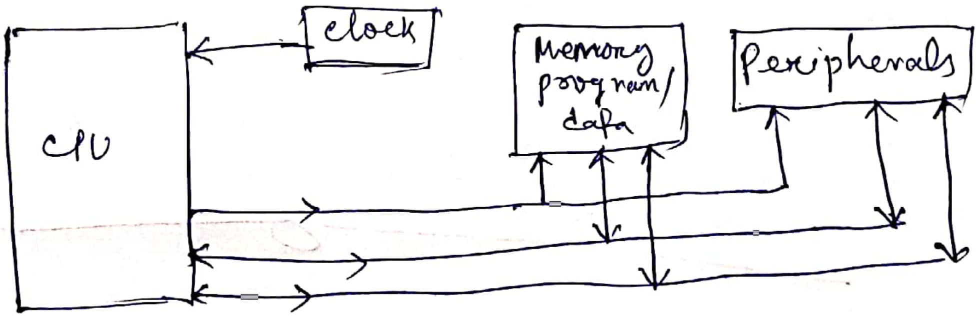
These are basically high speed microprocessors with architecture and instruction sets optimized for DSP operations. They include fixed point processors such as Texas instruments TMS320C5x, TMS320C54x & Motorola DSP563x & floating point processor such as Texas instruments TMS320C4x, & TMS320C67xx & analog device ADSP21xxx.

(ii) Special purpose digital signal processors :-

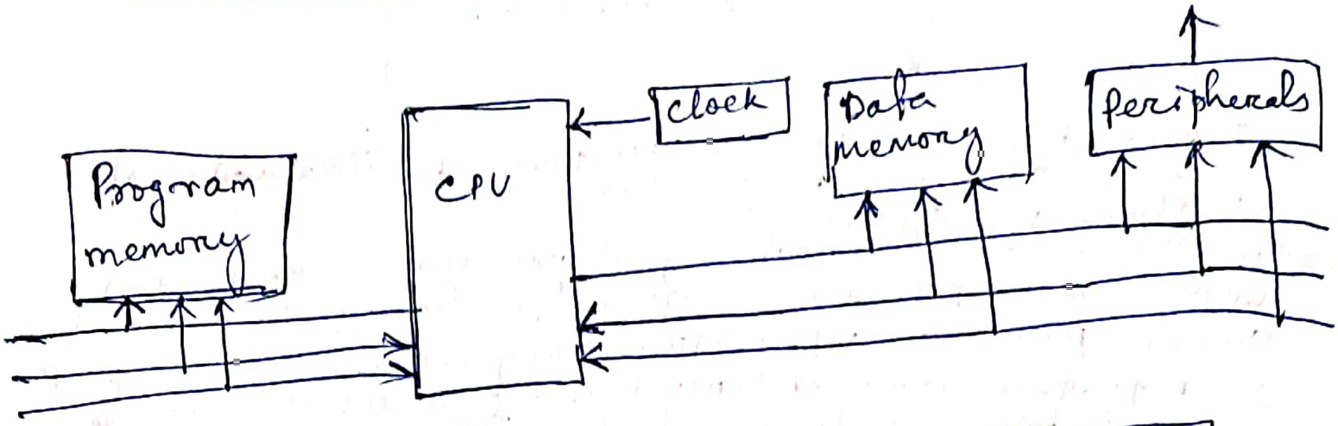
These type of processors consists of hardware i) designed for specific DSP algorithms such as FFT, (ii) hardware designed for specific applications such as PCM & filtering. Examples for special purpose DSPs are Mitel's multi channel telephony voice echo canceller (MT93001), FFT processor (PDSP 16515A, TM-44, TM-66) & programmable FIR filter (UPDSP 16256, model 3092).

Different types of Architecture :-

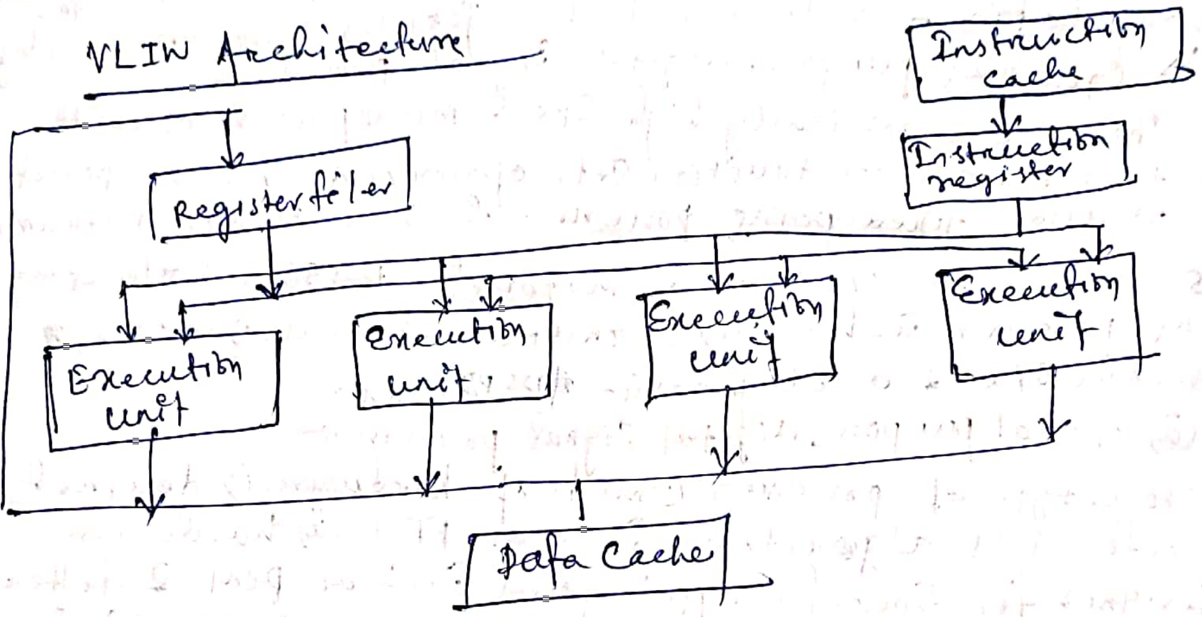
→ Von Neuman Architecture →



Harvard architecture



VLIW Architecture



VLIW (very long Instruction word)